Lie Algebras to Root Systems

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A Lie algebra is a vector space over some field with some binary operation that satisfies the following axioms:

- **Bilinearity**
  - \([ax + by, z] = a[x, z] + b[y, z]\)

- **Alternating**
  - \([x, x] = 0\)

- **Satisfies the Jacobi identity**
  - \([x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0\)
A root system over some finite-dimensional Euclidean vector space $V$ is a set of non-zero vectors (roots) which satisfy the following properties:

- The roots span $V$
- The only scalar multiples of a root $x$ appearing in the set are $x$ and $-x$.
- For every root, the set is closed under reflection through the hyperplane perpendicular to that root.
- $V$ must be an inner-product space.
- For roots $\alpha, \beta$, $\langle \alpha, \beta \rangle := 2(\beta, \alpha)/(\alpha, \alpha)$ is an integer.
Examples of Root Systems
Our Basis

\[ h_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad e_{12} = \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) \quad e_{21} = \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right) \]

\[ h_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad e_{13} = \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) \quad e_{31} = \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) \]

\[ e_{23} = \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) \quad e_{32} = \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) \]
Resources

- Jon Cohen
- *Introduction to Lie Algebras* by Karin Erdmann and Mark J. Wildon
- *Introduction to Lie Algebras and Representation Theory* by James E. Humphreys
- ...Mostly Jon Cohen