USING THE RANDOM ITERATION ALGORITHM TO CREATE FRACTALS

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THE SIERPINSKI GASKET

Stage 0:
\[ A_0 = \frac{1}{2} \sqrt{2}^2 \]
\[ A_0 = 1 \]

Stage 1:
\[ A_1 = 1 - \frac{1}{2} \left( \frac{\sqrt{2}}{2} \right)^2 \]
\[ A_1 = 1 - \frac{1}{4} \]
\[ A_1 = \frac{3}{4} \]

Stage 2:
\[ A_2 = \frac{3}{4} - \frac{3}{2} \left( \frac{\sqrt{2}}{4} \right)^2 \]
\[ A_2 = \frac{3}{4} - \frac{3}{16} = \frac{9}{16} \]

Stage n:
\[ A_n = \left( \frac{3}{4} \right)^n \]
\[ \lim_{n \to \infty} \left( \frac{3}{4} \right)^n = 0 \]

Sierpinski Gasket has zero area?
Let $S \subseteq \mathbb{R}^n$ where $n = 1, 2, \text{ or } 3$

$n$-dimensional box

- $n = 1$: Closed interval
- $n = 2$: Square
- $n = 3$: Cube

Let $N(\varepsilon) = \text{smallest number of } n\text{-dimensional boxes of side length } \varepsilon \text{ necessary to cover } S$
CAPACITY DIMENSION AND FRACTALS

\( n = 1 \)

Boxes of length \( \varepsilon \) to cover line of length \( L \):

If \( L = 10 \text{cm} \) and \( \varepsilon = 1 \text{cm} \), it takes 10 boxes to cover \( L \)

If \( \varepsilon = 0.5 \text{cm} \), it takes 20 boxes to cover \( L \)

\[ N(\varepsilon) \propto \frac{1}{\varepsilon} \]
CAPACITY DIMENSION AND FRACTALS

\( n = 2 \)

Boxes of length \( \varepsilon \) to cover square \( S \) of side length \( L \)

If \( L = 20 \text{cm} \), area of \( S = 400 \text{cm}^2 \)

- \( \varepsilon = 2 \text{cm} \): each box has area \( 4 \text{cm}^2 \)
  - It will take 100 boxes to cover \( S \)
- \( \varepsilon = 1 \text{cm} \): each box has area \( 1 \text{cm}^2 \)
  - It will take 400 boxes to cover \( S \)

\( N(\varepsilon) \propto \frac{1}{\varepsilon^2} \)
CAPACITY DIMENSION AND FRACTALS

\[ N(\varepsilon) = C \left( \frac{1}{\varepsilon} \right)^D \]

\[ \ln(N(\varepsilon)) = D \ln \left( \frac{1}{\varepsilon} \right) + \ln(C) \]

\[ D = \frac{\ln(N(\varepsilon)) - \ln(C)}{\ln \left( \frac{1}{\varepsilon} \right)} \quad \text{C just depends on scaling of S} \]

Capacity Dimension: \( \dim_c S = \lim_{\varepsilon \to 0^+} \frac{\ln(N(\varepsilon))}{\ln \left( \frac{1}{\varepsilon} \right)} \)
CAPACITY DIMENSION AND FRACTALS

Stage 1:
If $\varepsilon = \frac{1}{2}$, $N(\varepsilon) = 3$

Stage 2:
If $\varepsilon = \frac{1}{4}$, $N(\varepsilon) = 9$

Stage n:
If $\varepsilon = \frac{1}{2^n}$, $N(\varepsilon) = 3^n$

$$\lim_{\varepsilon \to 0^+} \frac{\ln(N(\varepsilon))}{\ln(\frac{1}{\varepsilon})} = \lim_{\varepsilon \to 0^+} \frac{\ln(3^n)}{\ln(2^n)} = \frac{n \ln(3)}{n \ln(2)} = \frac{\ln(3)}{\ln(2)} \approx 1.5849625$$
The Sierpinski Gasket is ≈ 1.585 dimensional.

A set with non-integer capacity dimension is called a **fractal**.
An Iterated Function System (IFS) $F$ is the union of the contractions $T_1, T_2 \ldots T_n$

**THEOREM:** Let $F$ be an iterated function system of contractions in $\mathbb{R}^2$. Then there exists a unique compact subset $A_F$ in $\mathbb{R}^2$ such that for any compact set $B$, the sequence of iterates $\{F^n(B)\}_{n=1}^{\infty}$ converges in the Hausdorff metric to $A_F$

$A_F$ is called the **attractor** of $F$.

This means that if we iterate any compact set in $\mathbb{R}^2$ under $F$, we will obtain a unique attractor (attractor depends on the contractions in $F$)
ITERATED FUNCTION SYSTEMS

A function $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is **affine** if it is in the form

$$f(x, y) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} ax + by + e \\ cx + dy + f \end{bmatrix}$$

(linear function followed by translation)

We will deal with iterated function systems of affine contractions.
RANDOM ITERATION ALGORITHM

Drawing an attractor of IFS $F$:
1. Choose an arbitrary initial point $\vec{v} \in \mathbb{R}^2$
2. Randomly select one of the contractions $T_n$ in $F$
3. Plot the point $T_n(\vec{v})$
4. Let $T_n(\vec{v})$ be the new $\vec{v}$
5. Repeat steps 2-4 to obtain a representation of $A_F$

1000 Iterations | 5000 Iterations | 20,000 Iterations
LET’S DRAW SOME FRACTALS!
ITERATIONS AND FIXED POINTS

Iterating = repeating the same procedure

Let $f(x)$ be a function.

- $f(f(x)) = f^2(x)$ is the second iterate of $x$ under $f$.
- $f(f(f((x)))) = f^3(x)$ is the third iterate of $x$ under $f$.

Example: $f(x) = x^2 + 1$

- $f(5) = 26$
- $f(f(5)) = f^2(5) = f(26) = 677$
ITERATIONS AND FIXED POINTS

A point $p$ is a **fixed point** for a function $f$ if its iterate is itself

$$f(p) = p$$

Example: $f(x) = x^2$

- $f(0) = 0$
- $f(f(0)) = f^2(0) = f(0) = 0$

Therefore 0 is a fixed point of $f$
METRIC SPACES

Let $S$ be a set. A **metric** is a distance function $d(x, y)$ that satisfies 4 axioms $\forall x, y \in S$

1. $d(x, y) \geq 0$
2. $d(x, y) = 0$ if and only if $x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, z) \leq d(x, y) + d(y, z)$

Example: Absolute Value $\mathbb{R}$

$d(x, y) = |x - y|$

$(\mathbb{R}, d)$ is a **metric space**
METRIC SPACES

Let \((S, d)\) be a metric space.

A sequence \(\{x_n\}_{n=1}^\infty\) in \(S\) **converges** to \(x \in S\) if \(\lim_{n \to \infty} d(x_n, x) = 0\)

This means that terms of the sequence approach a value \(s\)

A sequence is **Cauchy** if for all \(\varepsilon > 0\) there exists a positive integer \(N\) such that whenever \(n, m \geq N\), \(d(x_n, x_m) < \varepsilon\)

This means that terms of the sequence get closer together

\((S, d)\) is a **complete metric space** if every Cauchy sequence in \(S\) converges to a member of \(S\)
CONTRACTION MAPPING THEOREM

Let \((S, d)\) be a metric space.

A function \(T: S \rightarrow S\) is a **contraction** if \(\exists q \in [0,1)\) such that
\[
d(T(x), T(y)) \leq q \cdot d(x, y)
\]
**CONTRACTION MAPPING THEOREM**

**Contraction Mapping Theorem:**

If \((S, d)\) is a complete metric space, and \(T\) is a contraction, then as \(n \to \infty\), \(T^n(x)\) → unique fixed point \(x^* \ \forall \ x \in S\)
HAUSDORFF METRIC

A set $S$ is **closed** if whenever $x$ is the limit of a sequence of members of $T$, $x$ actually is in $T$.

A set $S$ is **bounded** if it there exists $x \in S$ and $r > 0$ such that $\forall s \in S$, $d(x, s) < r$

Means $S$ is contained by a “ball” of finite radius

A set $S \subseteq \mathbb{R}^n$ is **compact** if it is closed and bounded

Let $K$ denote all compact subsets of $\mathbb{R}^2$
HAUSDORFF METRIC

If $B$ is a nonempty member of $K$, and $\mathbf{v}$ is any point in $\mathbb{R}^2$, the distance from $\mathbf{v}$ to $B$ is

$$d(\mathbf{v}, B) = \text{minimum value of } \|\mathbf{v} - \mathbf{b}\| \quad \forall \mathbf{b} \in B$$

(distance from point to a compact set)
HAUSDORFF METRIC

If $A$ and $B$ are members of $K$, then the distance from $A$ to $B$ is

$$d(A, B) = \text{maximum value of } d(\tilde{a}, B) \text{ for } \tilde{a} \in A$$

(distance between compact sets)

Means we take the point in $A$ that is most distant from any point in $B$ and find the minimum distance between it an any point in $B$
The **Hausdorff metric** on $K$ is defined as:

$$D(A, B) = \text{maximum of } d(A, B) \text{ and } d(B, A)$$

**Diagram:**

- $A$ and $B$ are sets in $K$.
- $d(A, B)$ is the distance between sets $A$ and $B$.
- $d(B, A)$ is the distance between sets $B$ and $A$.

In the diagram, $D(A, B) = d(B, A)$. The Hausdorff distance measures the maximum distance that any point in one set must be from the closest point in the other set.
Iterated Function System:

\[ f_1(x, y) = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} (x, y) \]

\[ f_2(x, y) = \begin{bmatrix} -1/2 & -1/2 \\ 1/2 & -1/2 \end{bmatrix} (x, y) + [1] \]

Render Details:

Point size: 1px

# of iterations: 100,000
Iterated Function System:

\[ f_1(x, y) = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}(x, y) \]

\[ f_2(x, y) = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}(x, y) + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} \]

\[ f_3(x, y) = \begin{bmatrix} 0 & -1/2 \\ 1/2 & 0 \end{bmatrix}(x, y) + \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} \]

Render Details:

Point size: 1px
# of iterations: 100,000
Iterated Function System:

\[
\begin{align*}
  f_1(x, y) &= \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix} (x, y) + \begin{bmatrix} 2/3 \\ 2/3 \end{bmatrix} \\
  f_2(x, y) &= \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix} (x, y) + \begin{bmatrix} 2/3 \\ -2/3 \end{bmatrix} \\
  f_3(x, y) &= \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix} (x, y) + \begin{bmatrix} -2/3 \\ 2/3 \end{bmatrix} \\
  f_4(x, y) &= \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix} (x, y) + \begin{bmatrix} -2/3 \\ -2/3 \end{bmatrix} \\
  f_5(x, y) &= \begin{bmatrix} 13/40 & 13/40 \\ -13/40 & 13/40 \end{bmatrix} (x, y)
\end{align*}
\]

Render Details:
Point size: 1px
# of iterations: 100,000
Iterated Function System:

\[ f_1(x, y) = \begin{bmatrix} 0 & 0 \\ 0 & 0.16 \end{bmatrix} (x, y) \]

\[ f_2(x, y) = \begin{bmatrix} 0.85 & 0.04 \\ -0.04 & 0.85 \end{bmatrix} (x, y) + \begin{bmatrix} 0 \\ 1.6 \end{bmatrix} \]

\[ f_3(x, y) = \begin{bmatrix} 0.2 & -0.26 \\ 0.23 & 0.22 \end{bmatrix} (x, y) + \begin{bmatrix} 0 \\ 1.6 \end{bmatrix} \]

\[ f_4(x, y) = \begin{bmatrix} -0.15 & 0.28 \\ 0.26 & 0.24 \end{bmatrix} (x, y) + \begin{bmatrix} 0 \\ 0.44 \end{bmatrix} \]

Probabilities:

\[ f_1: 1\% \quad f_2: 85\% \quad f_3: 7\% \quad f_4: 7\% \]
Iterated Function System:

\[
\begin{align*}
    f_1(x) &= \begin{bmatrix} 1/2 & -\sqrt{3}/6 \\ \sqrt{3}/6 & 1/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
    f_2(x) &= \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1/\sqrt{3} \\ 1/3 \end{bmatrix} \\
    f_3(x) &= \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1/\sqrt{3} \\ -1/3 \end{bmatrix} \\
    f_4(x) &= \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1/\sqrt{3} \\ 1/3 \end{bmatrix} \\
    f_5(x) &= \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1/\sqrt{3} \\ -1/3 \end{bmatrix} \\
    f_6(x) &= \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 2/3 \end{bmatrix} \\
    f_7(x) &= \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -2/3 \end{bmatrix}
\end{align*}
\]
THANKS
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Some IFS Formulas from Agnes Scott College

Graphs Created with Desmos Graphing Calculator