Weak Solutions to Partial Differential Equations

Case study: Poisson’s Equation

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Outline

1. Introduction
2. Weak formulation
3. Functional Analysis
4. Existence and Uniqueness
5. Regularity
Motivation

Poisson’s Equation

For \( u \in C^2(U) \) we say that \( u \) satisfies Poisson’s equation if

\[
\Delta u = f \quad \text{on } U \subset \mathbb{R}^n
\]

\[
u = 0 \quad \text{on } \partial U
\]

Figure: A steady state of the heat equation
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Weak Derivatives

Definition

For, $\alpha$ a multiindex and $f \in L^p(U)$ for some open set $U$, we say $g = D^\alpha f$ is the weak derivative of $f$ if

$$\int_U g \phi \, dx = (-1)^{|\alpha|} \int_U f \, D^\alpha \phi \, dx$$

for each $\phi \in C_c^\infty(U)$. 

Example 1

Figure: Example of a weakly differentiable function $f(x) = |x|$

\[-\int_{-\infty}^{\infty} |x| \phi' \, dx = -\int_{-\infty}^{0} -x \phi' \, dx - \int_{0}^{\infty} x \phi' \, dx = \int_{-\infty}^{\infty} h(x) \phi \, dx\]

for each $\phi \in C_c^\infty(\mathbb{R})$ where $h(x)$ is the weak derivative of $f$ given by

\[h(x) = \begin{cases} 
1 & \text{if } x \geq 0 \\
-1 & \text{if } x < 0
\end{cases}\]
Example 2

\[ g(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \]

**Figure**: Example of a non-weakly differentiable function \( g(x) \) the Heavyside function

\[
- \int_{-\infty}^{\infty} g(x) \phi' \, dx = - \int_{0}^{\infty} \phi' \, dx = \phi(0) = \int_{-\infty}^{\infty} \delta_0 \phi \, dx
\]
Sobolev Spaces

- Define the Sobolev spaces

\[ W^{k,p}(U) = \{ f \in L^p(U) \mid D^\alpha f \in L^p(U) \forall |\alpha| < k \}. \]

which we norm with

\[ \| f \|_{W^{k,p}} = \left( \sum_{|\alpha| \leq k} \| D^\alpha f \|_{L^p}^p \right)^{1/p} = \left( \| f \|_{L^p}^p + \| \nabla f \|_{L^p}^p + \cdots + \| \nabla^k f \|_{L^p}^p \right)^{1/p} \]

- In particular we denote the Hilbert space

\[ W^{k,2}(U) = H^k(U) \]
Weak Formulation for Poisson’s Equation

- Now, for the classical formulation of Poisson’s equation we have

\[ \Delta u = f \]

- We multiply by a test function \( v \in H^1(U) \) and integrate by parts to get the weak formulation,

\[
\int_U v \Delta u \, dx = - \int_U \nabla v \cdot \nabla u \, dx = \int_U fv \, dx
\]

Definition

We say that \( u \) is a weak solution to Poisson’s equation if for all \( v \in H^1 \), it satisfies

\[
B[u, v] = - \int_U \nabla v \cdot \nabla u \, dx = \int_U fv \, dx = f(v)
\]
Energy Estimates/Lax Milgram

**Theorem**

Let $H$ be a Hilbert space, and $B : H \times H \rightarrow \mathbb{R}$ be a bilinear form satisfying the energy estimates for some constants $\alpha, \beta > 0$,

$$|B[u, v]| \leq \alpha \|u\|_H \|v\|_H$$

and

$$\beta \|u\|_H^2 \leq |B[u, u]|.$$

Then, if $f : H \rightarrow \mathbb{R}$ be a bounded linear functional, there exists a unique $u \in H$ such that

$$B[u, v] = f(v)$$

for each $v \in H$. 
Energy Estimates for Poisson’s Equation

- First, by Holder’s inequality
  
  \[ |B(u, v)| \leq \int_{U} |\nabla u| |\nabla v| \, dx \leq \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} \leq \|u\|_{H^1} \|v\|_{H^1} \]

- By a Poincare inequality, for some \( \beta > 0 \),
  
  \[ \|u\|_{L^2}^2 \leq \beta \|\nabla u\|_{L^2}^2 = \beta \int_{U} |\nabla u|^2 \, dx = \beta |B(u, u)| \]

  Thus,
  
  \[ \|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \leq (\beta + 1)|B(u, u)| \]
Existence and Uniqueness for Poisson’s Equation

Definition

We say \( u \) is a weak solution of Poisson’s equation if \( u \) satisfies

\[
B[u, v] = -\int_{\Omega} \nabla v \cdot \nabla u \, dx = \int_{\Omega} fv \, dx = f(v)
\]

for each \( v \in H^1 \).

- By Lax-Milgram, there exists a unique weak solution \( u \) to Poisson’s equation.
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Assume $u \in C^2(U)$ for $u$ our weak solution

Then, for each $v \in H^1$,

$$- \int_U \nabla u \cdot \nabla v \, dx = \int_U v \Delta u = \int_U f v \, dx$$

So, for each $v \in H^1$,

$$\int_U (\Delta u - f) v \, dx = 0$$

Therefore,

$$\Delta u = f \text{ a.e. on } U$$
References

[1] L. Evans
Partial Differential Equations.
American Mathematical Society (2010)