Weak Solutions to Partial Differential Equations

Case study: Poisson's Equation

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Outline

Motivation

Poisson's Equation

For $u \in C^2(U)$ we say that u satisfies Poisson's equation if

$$\begin{aligned} \Delta u &= f && \text{on } U \subset \mathbb{R}^n \\ u &= 0 && \text{on } \partial U \end{aligned}$$

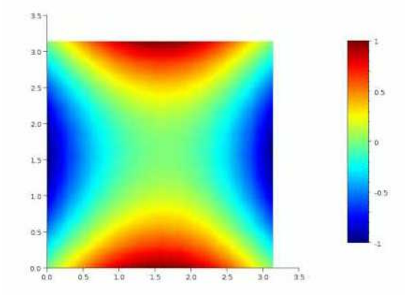

Figure : A steady state of the heat equation

Outline

Weak Derivatives

Definition

For, α a multiindex and $f \in L^p(U)$ for some open set U, we say $g = D^\alpha f$ is the weak derivative of f if

$$\int_U g\phi \, dx = (-1)^{|\alpha|} \int_U f \, D^\alpha \phi \, dx$$

for each $\phi \in C_c^\infty(U)$.

Example 1

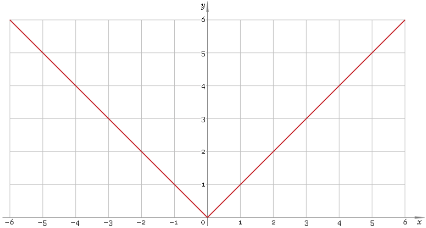

Figure : Example of a weakly differentiable function $f(x) = |x|$

$$-\int_{-\infty}^{\infty} |x|\phi' \, dx = -\int_{-\infty}^{0} -x\phi' \, dx - \int_{0}^{\infty} x\phi' \, dx = \int_{-\infty}^{\infty} h(x)\phi \, dx$$

for each $\phi \in C_c^\infty(\mathbb{R})$ where $h(x)$ is the weak derivative of f given by

$$h(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Example 2

$$g(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

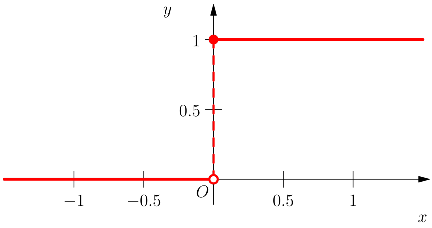

Figure : Example of a non-weakly differentiable function $g(x)$ the Heavyside function

$$-\int_{-\infty}^{\infty} g(x)\phi' \; dx = -\int_{0}^{\infty} \phi' \; dx = \phi(0) = \int_{-\infty}^{\infty} \delta_0 \phi \; dx$$

Sobolev Spaces

- Define the Sobolev spaces

$$W^{k,p}(U) = \{f \in L^p(U) \mid D^\alpha f \in L^p(U) \ \forall |\alpha| < k\}.$$

which we norm with

$$\|f\|_{W^{k,p}} = \left(\sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^p}^p \right)^{1/p} = \left(\|f\|_{L^p}^p + \|\nabla f\|_{L^p}^p + \cdots + \|\nabla^k f\|_{L^p}^p \right)^{1/p}$$

- In particular we denote the Hilbert space

$$W^{k,2}(U) = H^k(U)$$

Weak Formulation for Poisson's Equation

- Now, for the classical formulation of Poisson's equation we have

$$\Delta u = f$$

- We multiply by a test function $v \in H^1(U)$ and integrate by parts to get the weak formulation,

$$\int_U v\Delta u \; dx = -\int_U \nabla v \cdot \nabla u \; dx = \int_U fv \; dx$$

Definition

We say that u is a weak solution to Poisson's equation if $\forall v \in H^1$, it satisfies

$$B[u, v] = -\int_U \nabla v \cdot \nabla u \; dx = \int_U fv \; dx = f(v)$$

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Energy Estimates/Lax Milgram

Theorem

Let H be a Hilbert space, and $B : H \times H \to \mathbb{R}$ be a bilinear form satisfying the energy estimates for some constants $\alpha, \beta > 0$,

$$|B[u, v]| \leq \alpha \|u\|_H \|v\|_H$$

and

$$\beta \|u\|_H^2 \leq |B[u, u]|.$$

Then, if $f : H \to \mathbb{R}$ be a bounded linear functional, there exists a unique $u \in H$ such that

$$B[u, v] = f(v)$$

for each $v \in H$.

Energy Estimates for Poisson's Equation

- First, by Holder's inequality

$$|B(u,v)| \leq \int_U |\nabla u||\nabla v| \, dx \leq \|\nabla u\|_{L^2}\|\nabla v\|_{L^2} \leq \|u\|_{H^1}\|v\|_{H^1}$$

- By a Poincare inequality, for some $\beta > 0$,

$$\|u\|_{L^2}^2 \leq \beta\|\nabla u\|_{L^2}^2 = \beta \int_U |\nabla u|^2 \, dx = \beta|B(u,u)|$$

Thus,

$$\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \leq (\beta+1)|B(u,u)|$$

Outline

Existence and Uniqueness for Poisson's Equation

Definition

We say u is a weak solution of Poisson's equation if u satisfies

$$B[u,v] = -\int_U \nabla v \cdot \nabla u \; dx = \int_U fv \; dx = f(v)$$

for each $v \in H^1$.

- By Lax-Milgram, there exists a unique weak solution u to Poisson's equation.

Outline

Recovering Classical Solutions

- Assume $u \in C^2(U)$ for u our weak solution
- Then, for each $v \in H^1$,

$$-\int_U \nabla u \cdot \nabla v \; dx = \int_U v \Delta u = \int_U fv \; dx$$

- So, for each $v \in H^1$,

$$\int_U (\Delta u - f)v \; dx = 0$$

- Therefore,

$$\Delta u = f \text{ a.e. on } U$$

References

[1] L. Evans
Partial Differential Equations.
American Mathematical Society (2010)