Differential Geometry: Curvature, Maps, and Pizza

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What is Differential Geometry and Curvature?

- Differential Geometry studies the properties of curves and surfaces, and their higher dimensional analogs.

\[ \kappa_g(t) = \frac{x'(t)y''(t) - x''(t)y'(t)}{(x'(t)^2 + y'(t)^2)^{3/2}} \]

In general, curvature of a curve can be described by the reciprocal of the radius of the closest approximating circle to the curve.

\[ \kappa_g(t) = \frac{1}{R(t)} \]

Figure 1: Curvature can be measured through osculating circles.
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Given the curvature function $\kappa_g(t)$, there exists a regular curve parametrized by arc length $\vec{x}: I \to \mathbb{R}^2$ that has $\kappa_g(t)$ as its curvature function. Furthermore, the curve is uniquely determined up to a rigid motion in the plane.

In other words, if you have the curvature function of a planar curve, you can work backwards to parametrize the curve.

<table>
<thead>
<tr>
<th>Curvature</th>
<th>Curve</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Line</td>
</tr>
<tr>
<td>1</td>
<td>Unit Circle</td>
</tr>
<tr>
<td>$\frac{1}{(1+t^2)^{3/2}}$</td>
<td>Parabola</td>
</tr>
</tbody>
</table>

Table 1: Examples of curves and their curvatures.
Principal Curvature

- At every point on a surface, there are two normal vectors, we chose one and declare it to be the positive direction.
- Sectional curvature is created using the chosen normal vector and the tangent vector at each point.

![Diagram of principal curvature](image)

Figure 2: An infinite amount of sections are created.
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Infinite amount of normal sections determine the curvature function.

Out of all the sectional curvatures, there is a $\kappa_{\text{min}}$ and a $\kappa_{\text{max}}$.

The directions of the planes created by $\kappa_{\text{min}}$ and $\kappa_{\text{max}}$ are called the principal directions.

**Figure 2:** An infinite amount of sections are created.
Gaussian Curvature

- Gaussian Curvature is calculated by the product of the principal curvatures. $K = \kappa_{\text{min}}\kappa_{\text{max}}$. 

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![Figure 3: Positive, negative, and zero curvature respectively](image.png)
Gaussian Curvature Continued

Sphere
- \( K = \kappa_{\text{min}} \kappa_{\text{max}} = \frac{1}{r^2} > 0 \)

Hyperbolic Paraboloid
- \( K = \kappa_{\text{min}} \kappa_{\text{max}} = \frac{-1}{r^2} < 0 \)

Cylinder
- \( K = \kappa_{\text{min}} \kappa_{\text{max}} = 0 \cdot \kappa_{\text{min}} = 0 \)

Figure 4: One-Sheeted Hyperbolic Paraboloid has negative curvature.
Figure 5: Maps distort distance due to having no curvature
Applications of Gaussian Curvature

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Figure 6: Gaussian Curvature allows us to hold pizza correctly