Differential Geometry: Curvature, Maps, and Pizza

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    - In general, curvature of a curve can be described by the reciprocal of the radius of the closest approximating circle to the curve. $\kappa_g = \frac{1}{R(t)}$

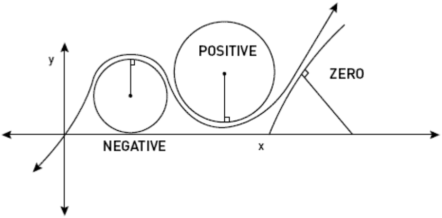

Figure 1: Curvature can be measured through osculating circles.

Fundamental Theorem of Planar Curves

- Given the curvature function $\kappa_g(t)$, there exists a regular curve parametrized by arc length $\vec{x} : I \to \mathbb{R}^2$ that has $\kappa_g(t)$ as its curvature function. Furthermore, the curve is uniquely determined up to a rigid motion in the plane.
- In other words, if you have the curvature function of a planar curve, you can work backwards to parametrize the curve

Curvature	Curve
0	Line
1	Unit Circle
$\frac{1}{(1+t^2)^{3/2}}$	Parabola

Table 1: Examples of curves and their curvatures.

Principal Curvature

- At every point on a surface, there are two normal vectors, we chose one and declare it to be the positive direction.
- Sectional curvature is created using the chosen normal vector and the tangent vector at each point

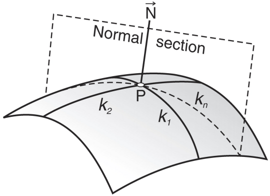

Figure 2: An infinite amount of sections are created.

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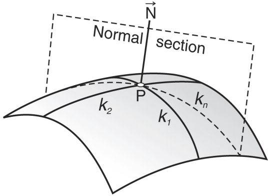

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- Infinite amount of normal sections determine the curvature function
- Out of all the sectional curvatures, there is a κ_{min} and a κ_{max}
- The directions of the planes created by κ_{min} and κ_{max} are called the principal directions.

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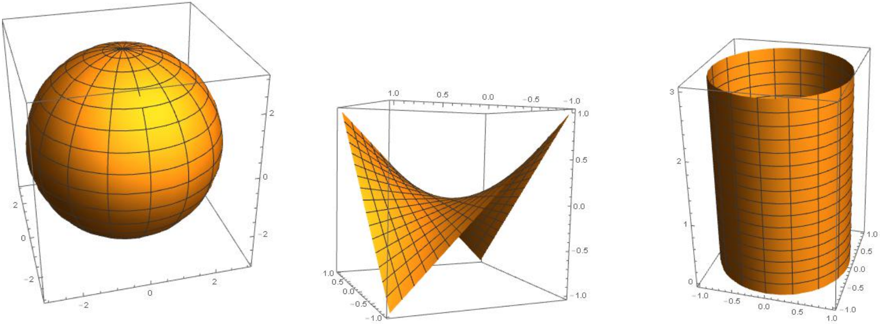

Figure 3: Positive, negative, and zero curvature respectively

Gaussian Curvature Continued

Sphere

- $K = \kappa_{\text{min}}\kappa_{\text{max}} = \frac{1}{r^2} > 0$

Hyperbolic Paraboloid

- $K = \kappa_{\text{min}}\kappa_{\text{max}} = \frac{-1}{r^2} < 0$

Cylinder

- $K = \kappa_{\text{min}}\kappa_{\text{max}} = 0 \cdot \kappa_{\text{min}} = 0$

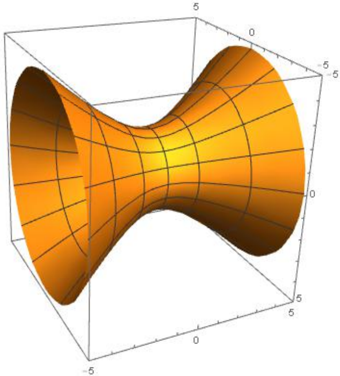

Figure 4: One-Sheeted Hyperbolic Paraboloid has negative curvature.

Applications of Gaussian Curvature

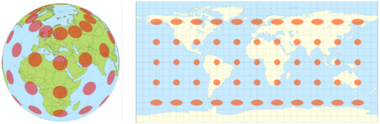

Figure 5: Maps distort distance due to having no curvature

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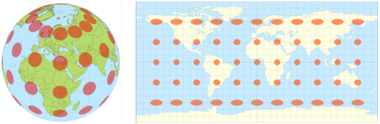

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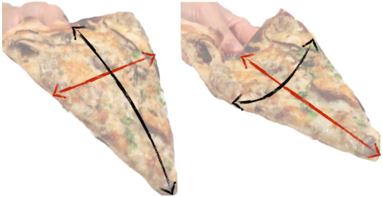

Figure 6: Gaussian Curvature allows us to hold pizza correctly