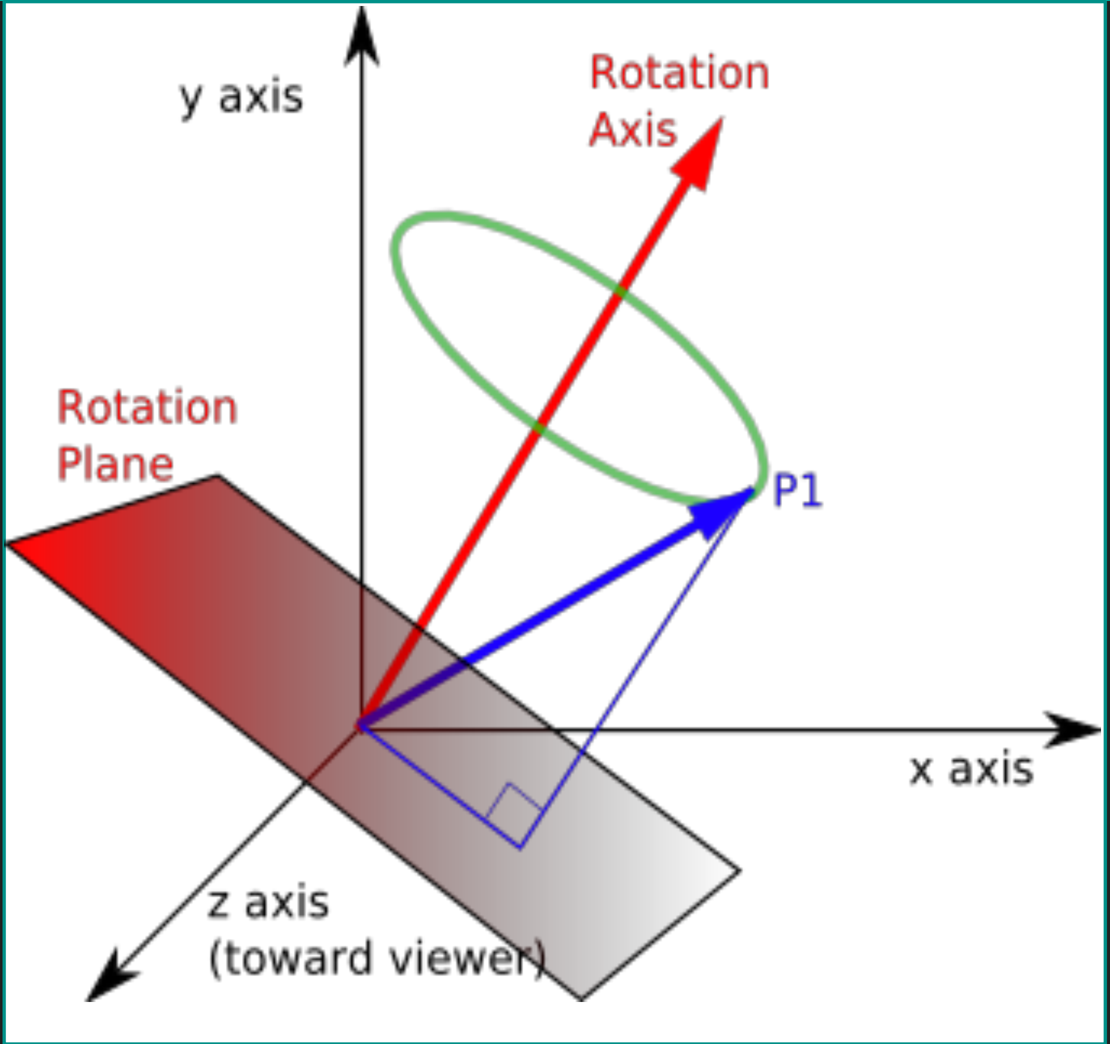
Rotations in 3D using Geometric Algebra

Kusal de Alwis

Mentor: Laura Iosip

Directed Reading Program, Spring 2019

The Problem



y axis
Rotation Axis
Rotation Plane
P1
x axis
z axis
(toward viewer)

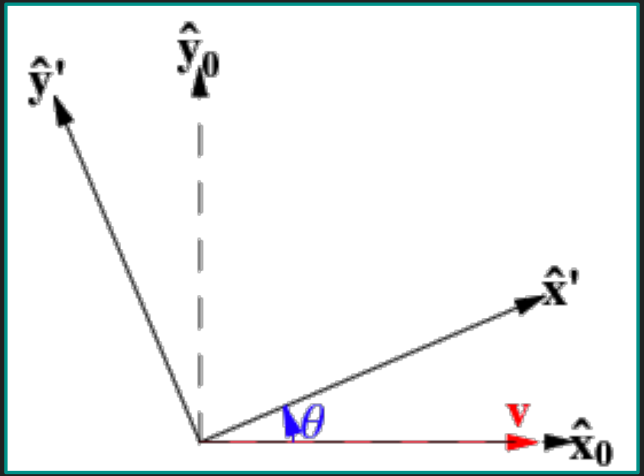
Overview

- Prior Methods for Rotations
- What is the Geometric Algebra?
- Rotating with Geometric Algebra
- Further Applications

Prior Methods

Rotation Matrix - 2D

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



Rotation Matrix - 3D

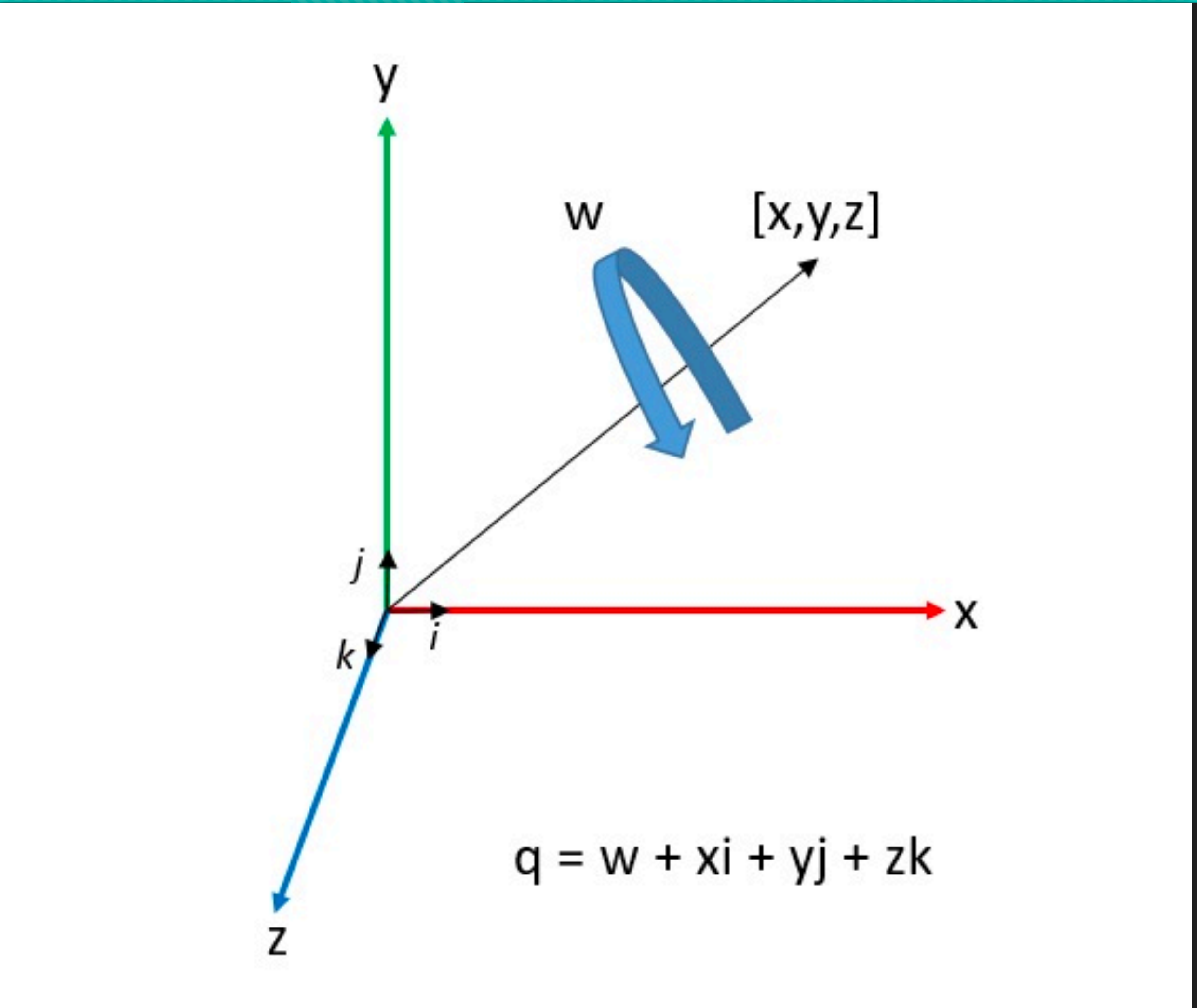
$$\begin{bmatrix} \cos\theta + u_x^2\left(1-\cos\theta\right) & u_x u_y\left(1-\cos\theta\right) - u_z\sin\theta & u_x u_z\left(1-\cos\theta\right) + u_y\sin\theta \\ u_y u_x\left(1-\cos\theta\right) + u_z\sin\theta & \cos\theta + u_y^2\left(1-\cos\theta\right) & u_y u_z\left(1-\cos\theta\right) - u_x\sin\theta \\ u_z u_x\left(1-\cos\theta\right) - u_y\sin\theta & u_z u_y\left(1-\cos\theta\right) + u_x\sin\theta & \cos\theta + u_z^2\left(1-\cos\theta\right) \end{bmatrix}$$

Rotation Matrix - 3D

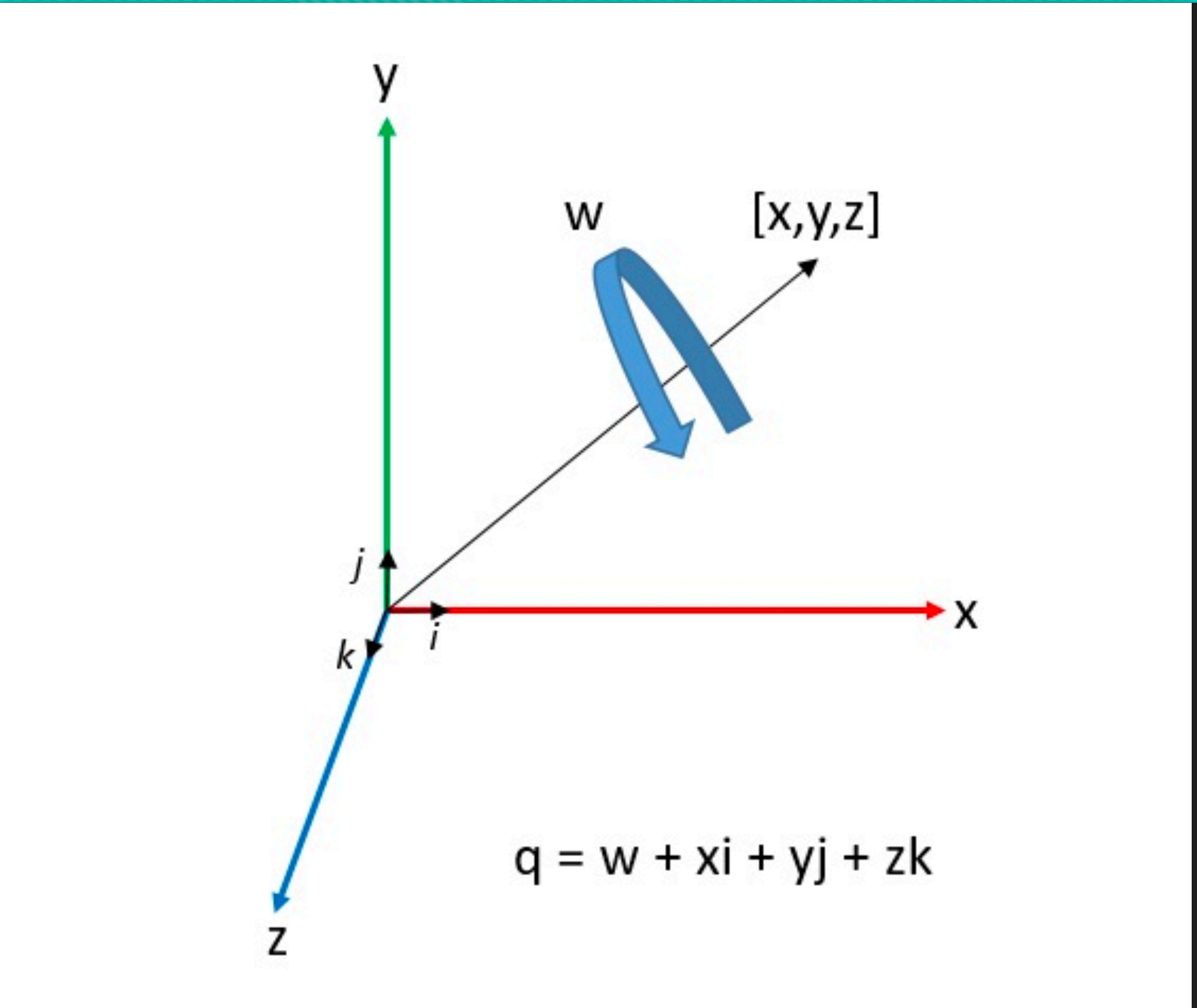
$$\begin{bmatrix} \cos\theta + u_x^2\left(1-\cos\theta\right) & u_x u_y\left(1-\cos\theta\right) - u_z\sin\theta & u_x u_z\left(1-\cos\theta\right) + u_y\sin\theta \\ u_y u_x\left(1-\cos\theta\right) + u_z\sin\theta & \cos\theta + u_y^2\left(1-\cos\theta\right) & u_y u_z\left(1-\cos\theta\right) - u_x\sin\theta \\ u_z u_x\left(1-\cos\theta\right) - u_y\sin\theta & u_z u_y\left(1-\cos\theta\right) + u_x\sin\theta & \cos\theta + u_z^2\left(1-\cos\theta\right) \end{bmatrix}$$

$$\begin{bmatrix} 0.36 & 0.48 & -0.8 \\ -0.8 & 0.60 & 0 \\ 0.48 & 0.64 & 0.60 \end{bmatrix}$$

Quaternions – 3D

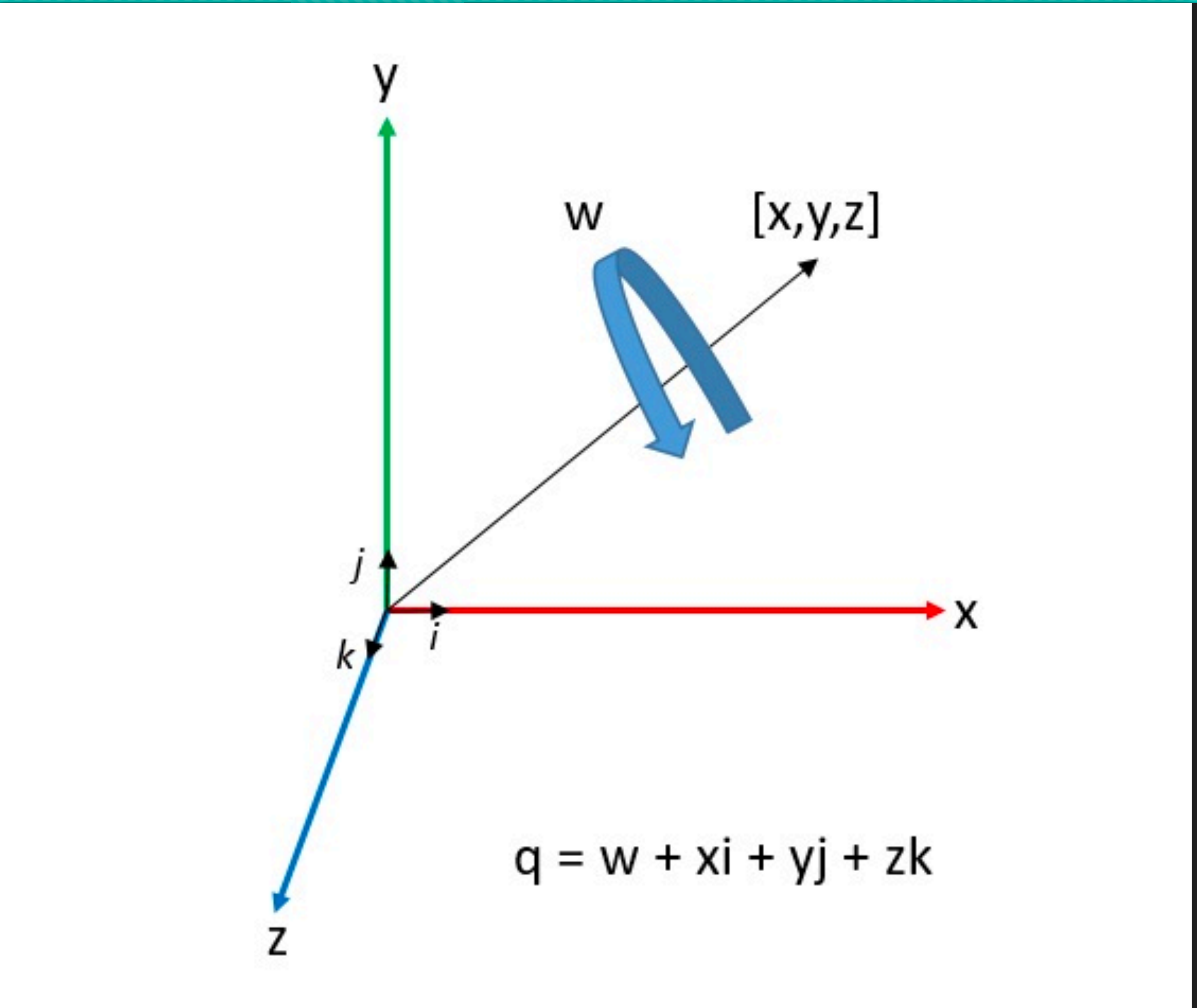


Quaternions – 3D



$$L_q(\boldsymbol{v}) = q\boldsymbol{v}q^*$$

Quaternions – 3D



$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

$$L_q(\boldsymbol{v}) = q\boldsymbol{v}q^*$$

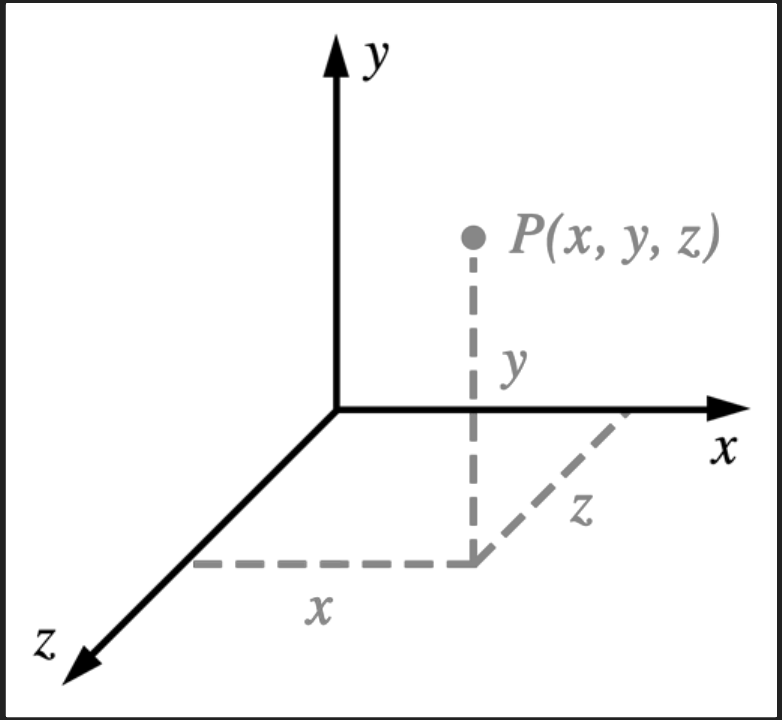
Octonions – 4D

		e_j							
	$e_i e_j$	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_i	e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
	e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
	e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
	e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
	e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
	e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
	e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
	e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

The Geometric Algebra

Extension of R^n

- Vectors and Scalars
- Same old algebra
  - Scalar Multiplication, Addition
- Only **one** augmentation…

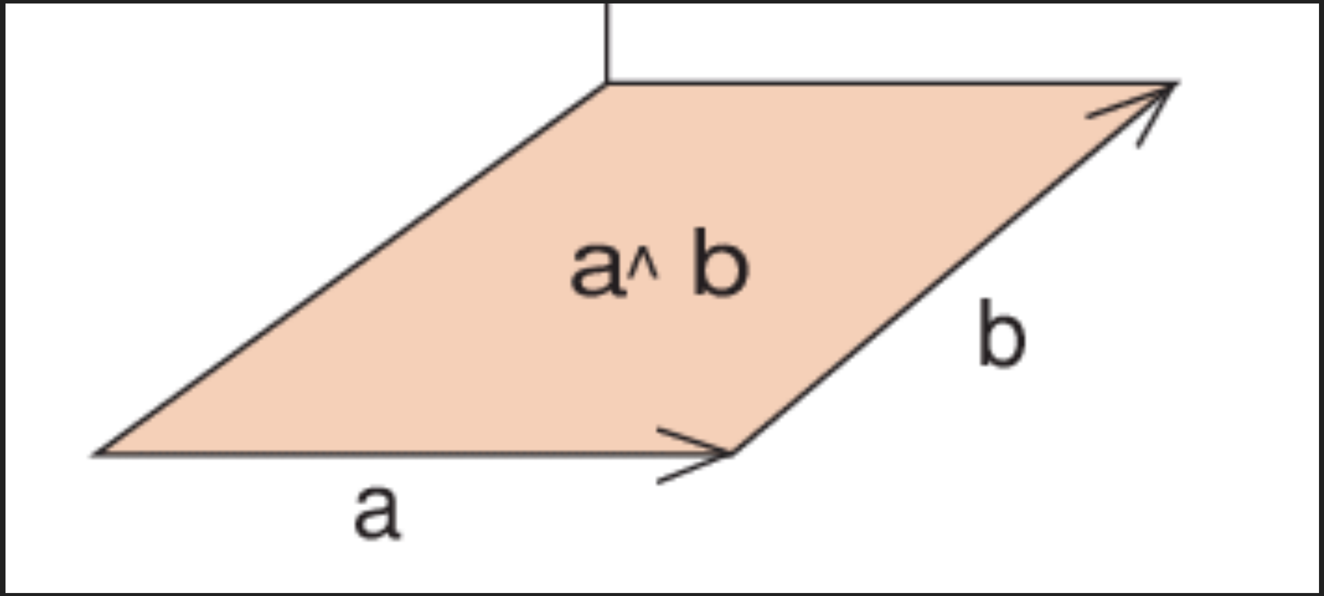


The Geometric Product

- Inner Product
  - Standard Dot Product

The Geometric Product

- Inner Product
  - Standard Dot Product
- Exterior Product
  - Another multiplication scheme
  - Represents planes



The Geometric Product

- Inner Product
  - Standard Dot Product
- Exterior Product
  - Another multiplication scheme
  - Represents planes
- Geometric Product

$$ab = a \cdot b + a \wedge b$$

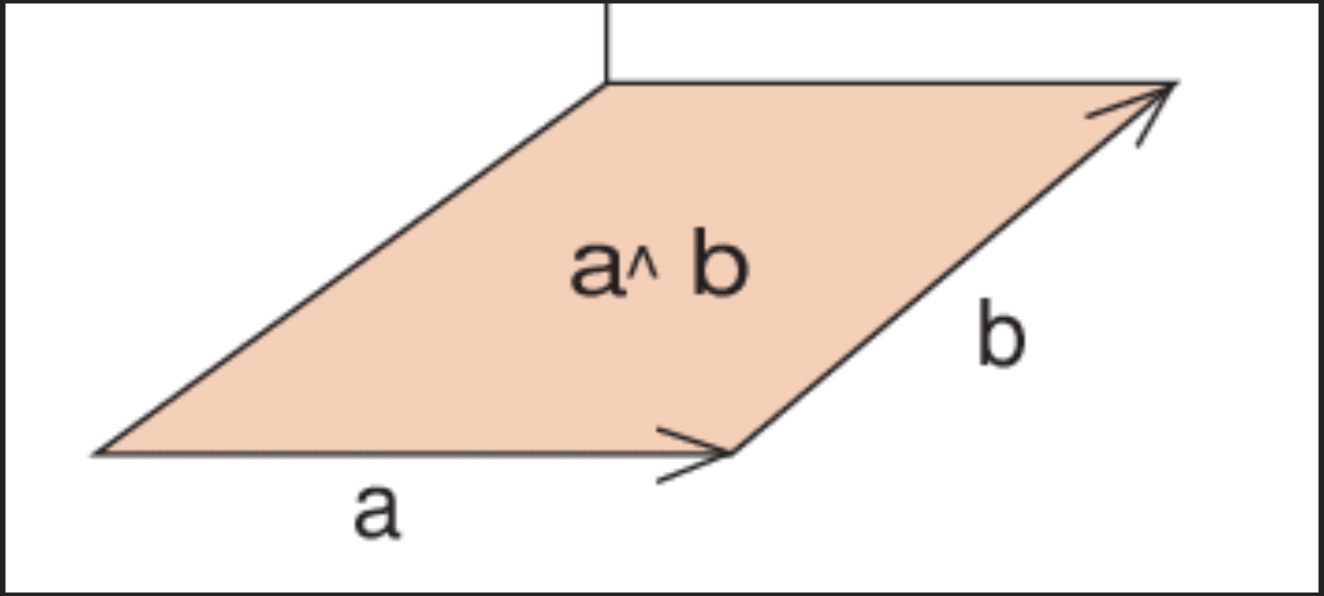
*Only for vectors

Implications of the Geometric Product

$$e_1 e_2 = e_1 \cdot e_2 + e_1 \wedge e_2 = 0 + e_1 \wedge e_2 = e_1 \wedge e_2$$

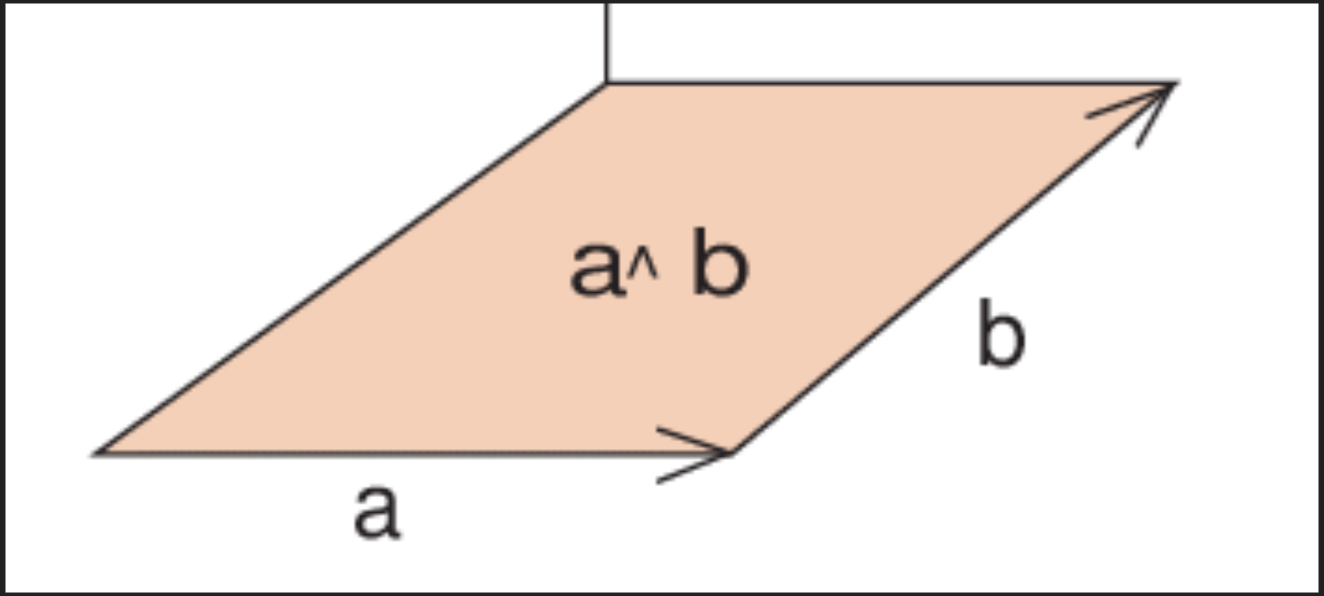
Implications of the Geometric Product

$$e_1 e_2 = e_1 \cdot e_2 + e_1 \wedge e_2 = 0 + e_1 \wedge e_2 = e_1 \wedge e_2$$



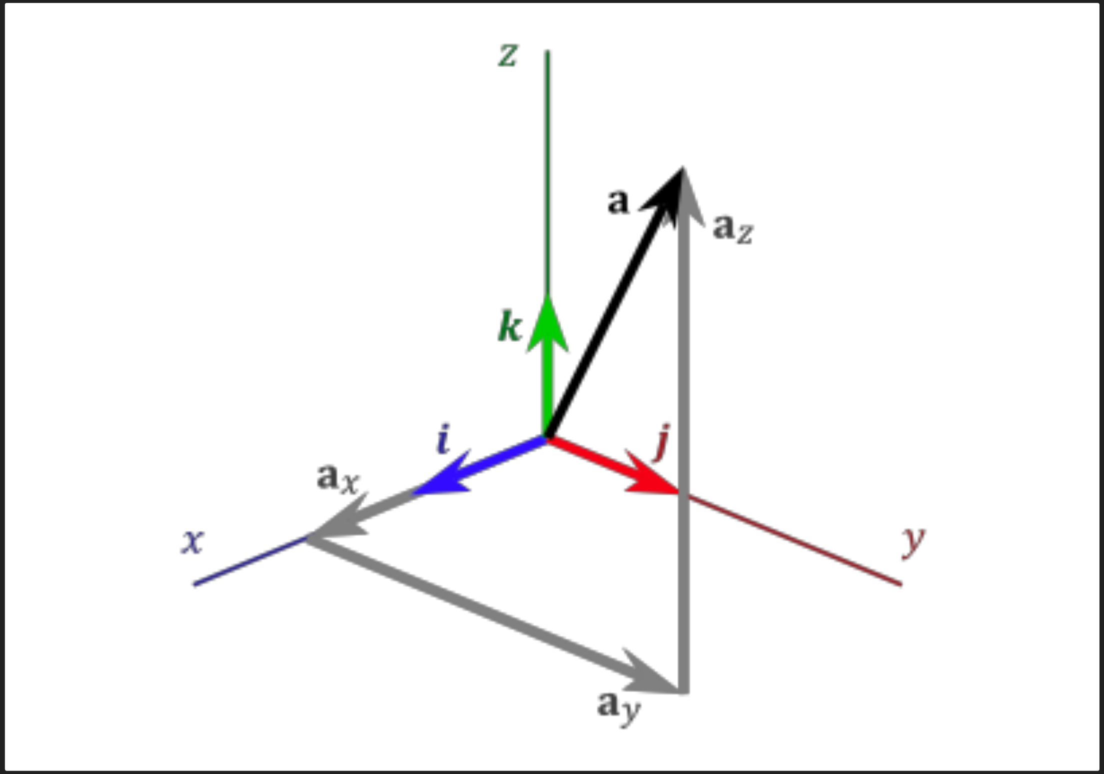
Implications of the Geometric Product

$$e_1 e_2 = e_1 \cdot e_2 + e_1 \wedge e_2 = 0 + e_1 \wedge e_2 = e_1 \wedge e_2$$

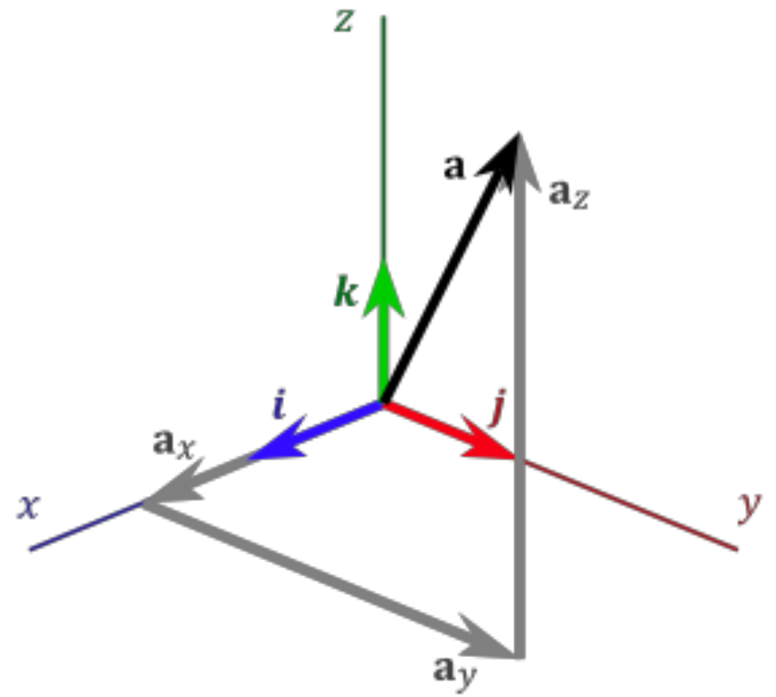


$$e_1(e_1 + e_2) = e_1 e_1 + e_1 e_2 = (e_1 \cdot e_1 + e_1 \wedge e_1) + (e_1 \cdot e_2 + e_1 \wedge e_2) = 1 + e_1 \wedge e_2$$

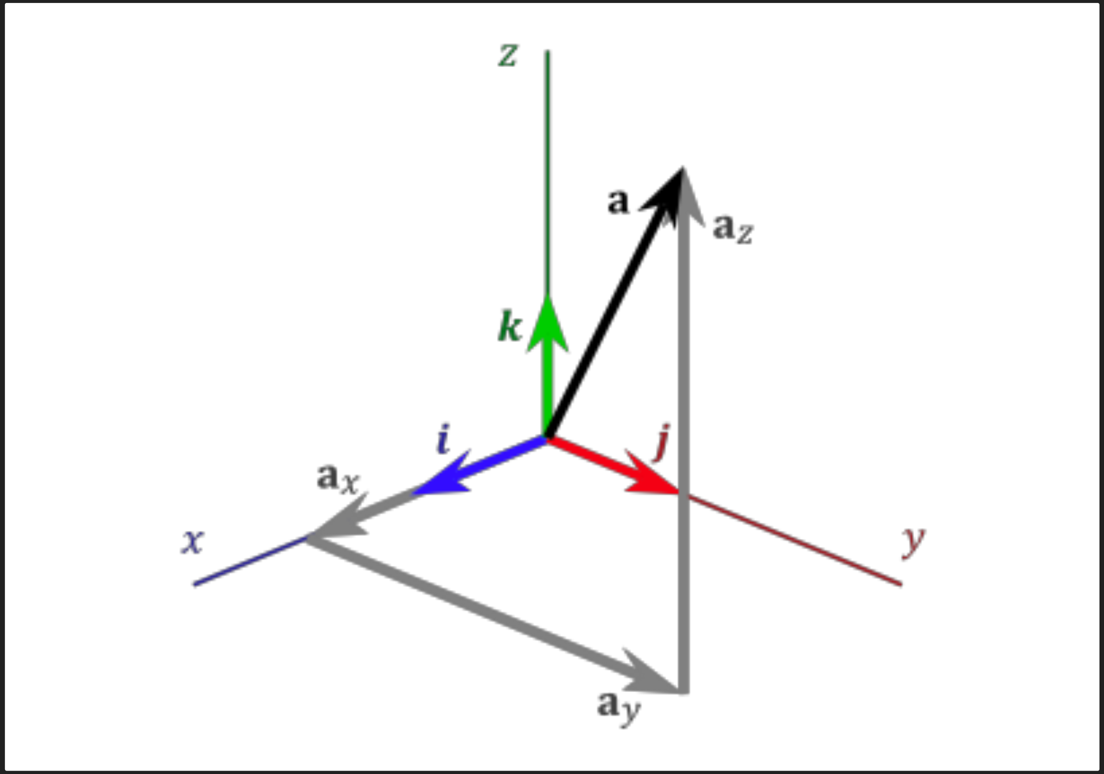
Multivectors in G^3

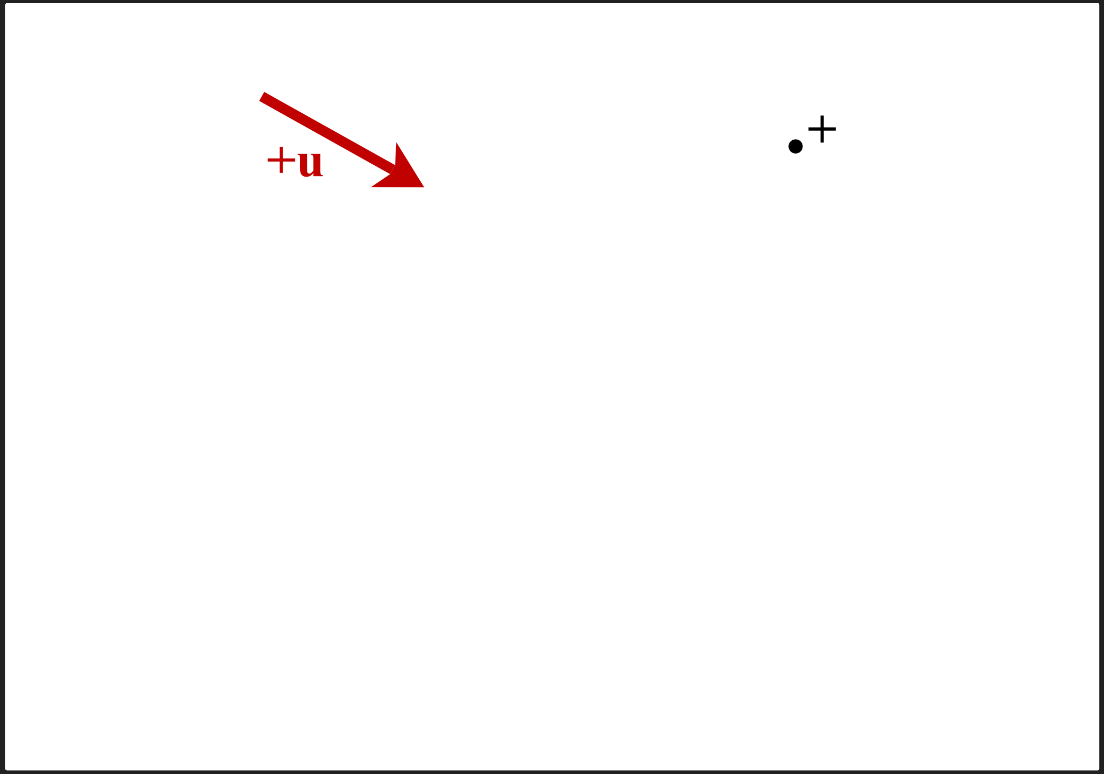


Multivectors in G^3

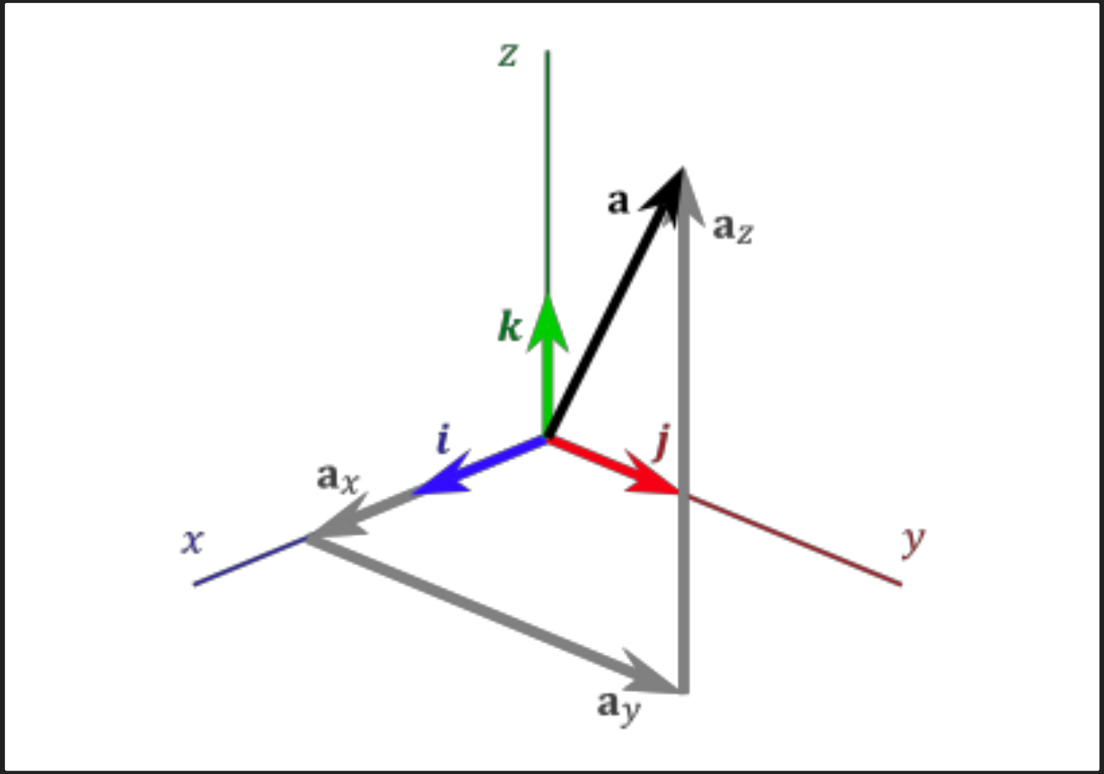


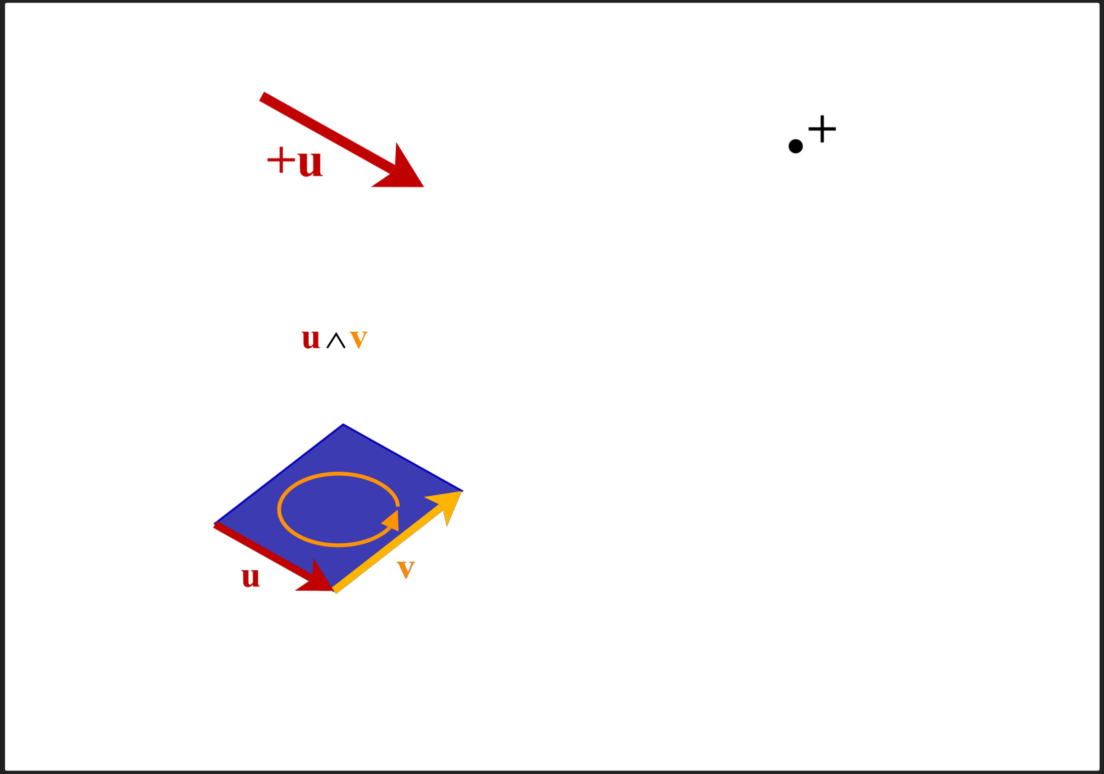
Multivectors in G^3



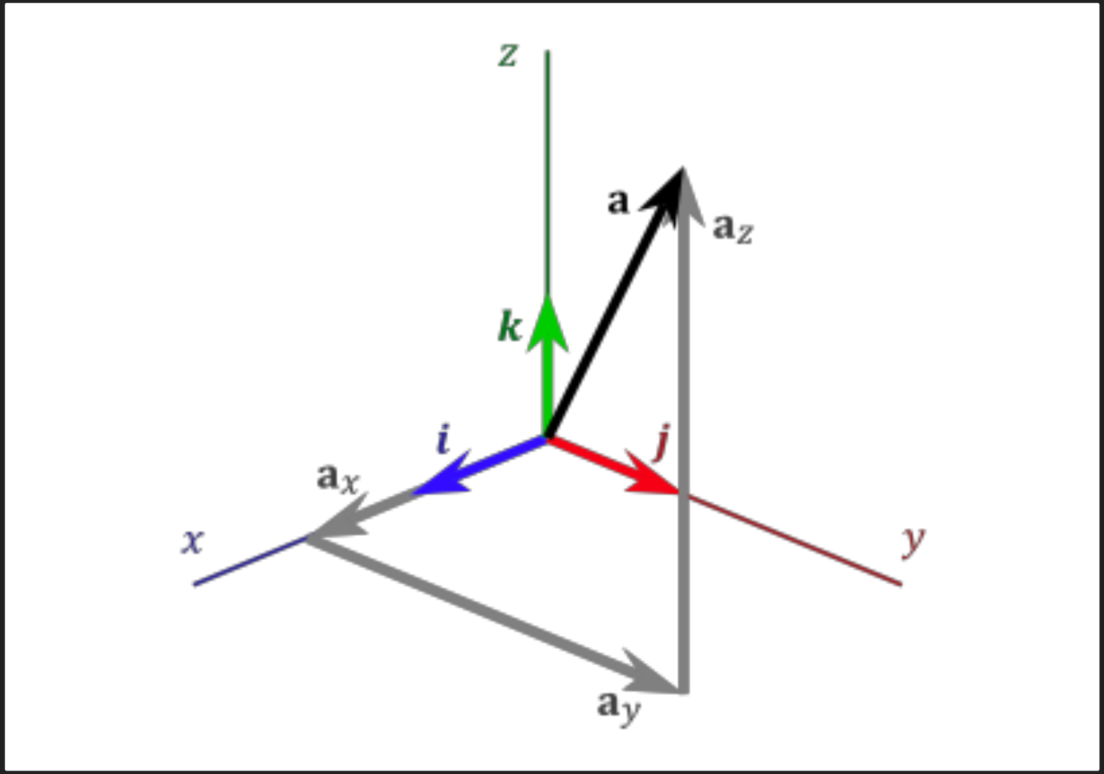


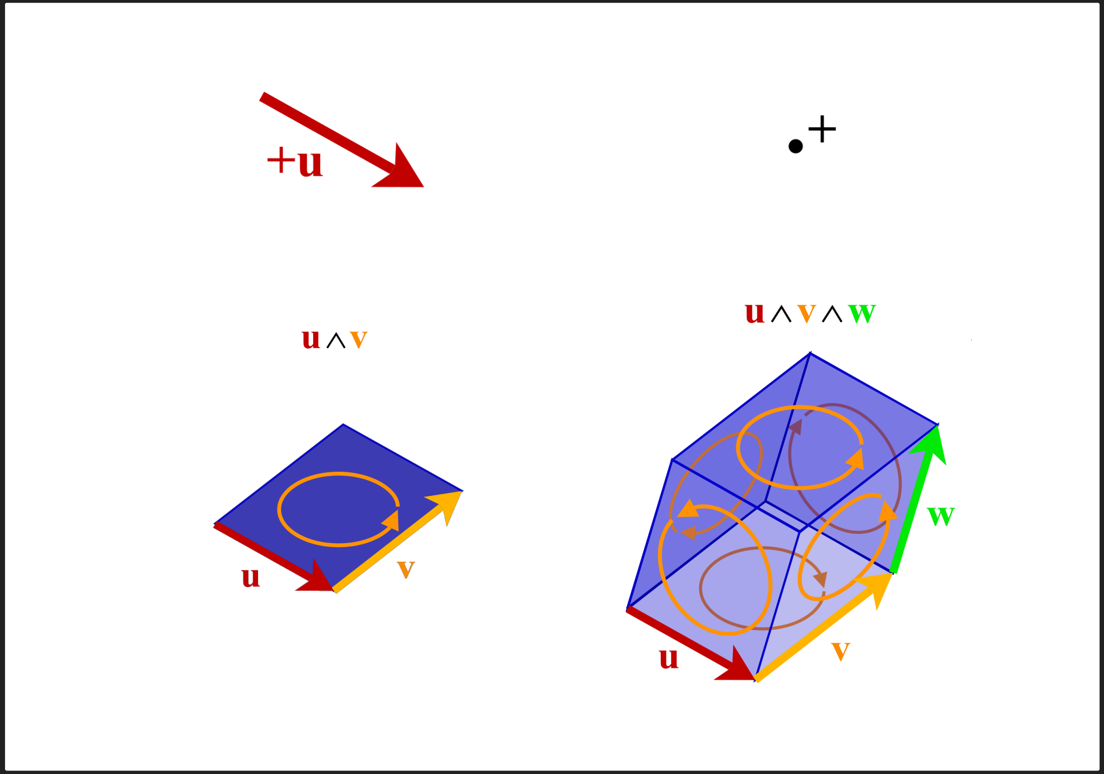
Multivectors in G^3





Multivectors in G^3





Basis of G^n

R^n

- e_1
- e_2
- e_3
- e_4
- $\ldots$
- e_n

- *n*-dimensional space

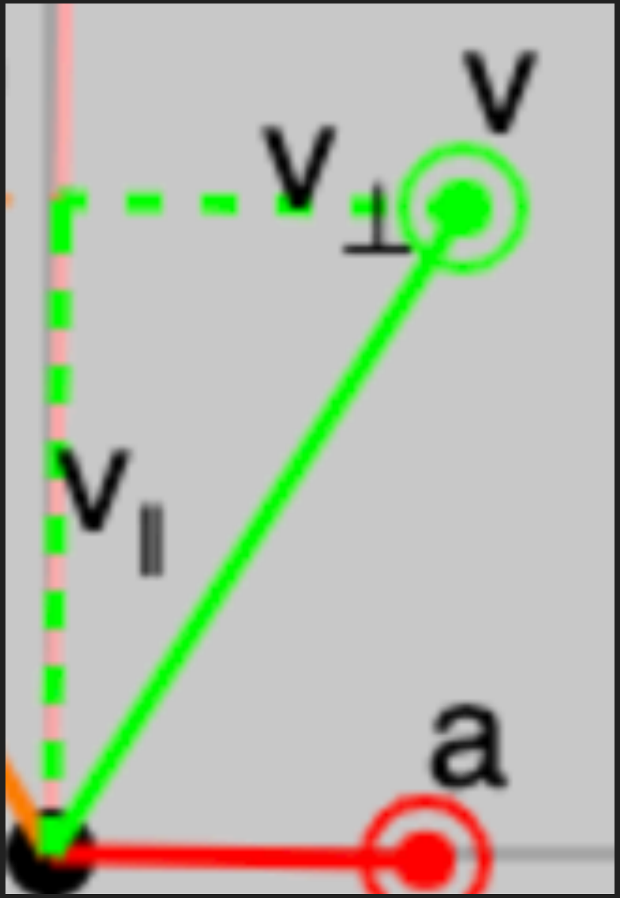
Basis of G^n

R^n	G^n
e_1	1
e_2	$e_1, e_2, e_3, \dots, e_n$
e_3	$e_1 \wedge e_2, e_1 \wedge e_3, \dots e_1 \wedge e_n, e_2 \wedge e_3, e_2 \wedge e_4, \dots e_2 \wedge e_n \dots e_{n-1} \wedge e_n$
e_4	$e_1 \wedge e_2 \wedge e_3, e_1 \wedge e_2 \wedge e_4, \dots$
$\dots$	$\dots$
e_n	$e_1 \wedge e_2 \wedge e_3 \wedge e_4 \wedge \cdots \wedge e_{n-1} \wedge e_n$
n-dimensional space	2^n-dimensional space

Rotations in G^n

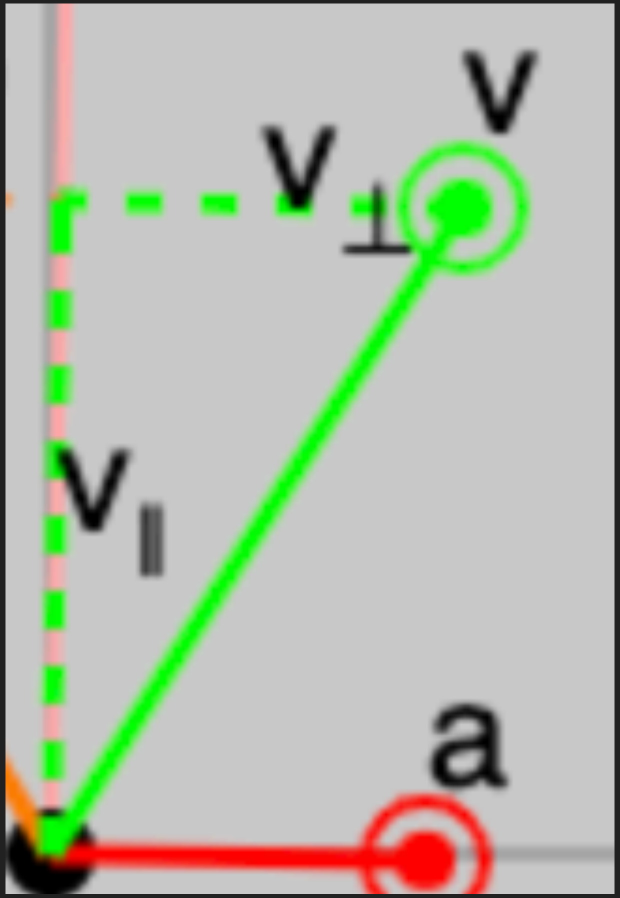
Projections & Rejections

- Unit a normal to a plane, Arbitrary v



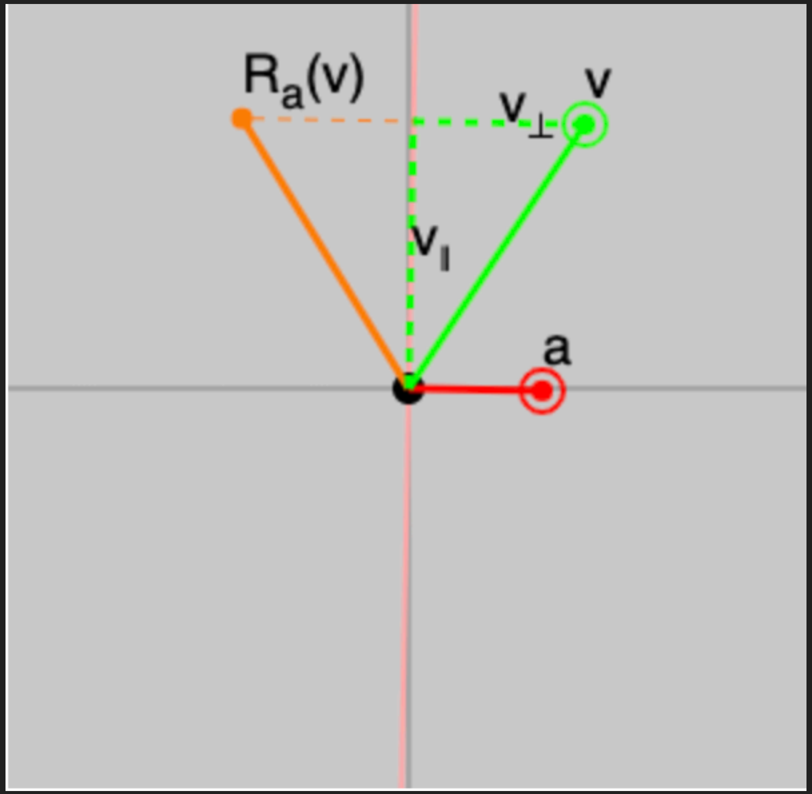
Projections & Rejections

- Unit a normal to a plane, Arbitrary v
- Projection
  - $v_{\perp} = (a \cdot v)a$
- Rejection
  - $v_{\parallel} = v - v_{\perp} = v - (a \cdot v)a$



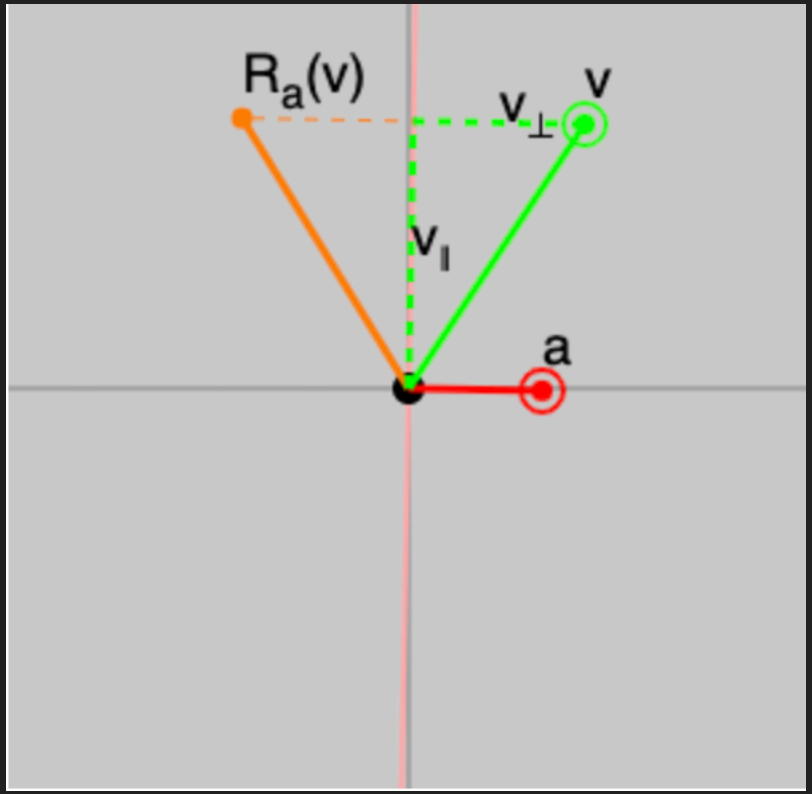
Reflection

- Reflect v on plane perpendicular to a
- $v = v_{\parallel} + v_{\perp}$



Reflection

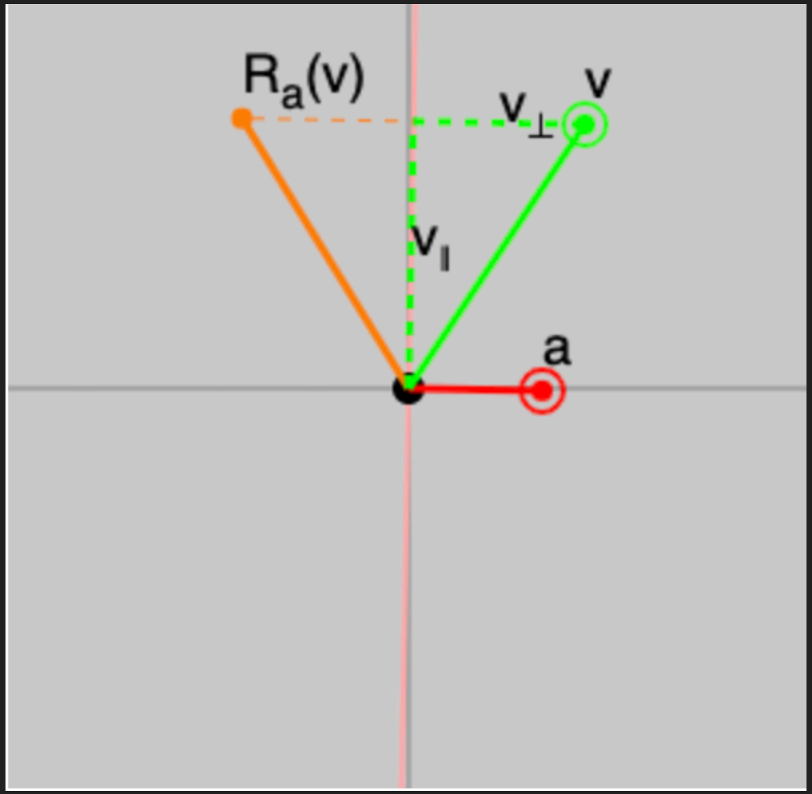
- Reflect v on plane perpendicular to a
- $v = v_{\parallel} + v_{\perp}$
- $R_a(v) = v_{\parallel} - v_{\perp}$



Reflection

- Reflect v on plane perpendicular to a
- $v = v_{\parallel} + v_{\perp}$
- $R_a(v) = v_{\parallel} - v_{\perp}$

$$a \cdot v = \frac{1}{2}(av + va)$$

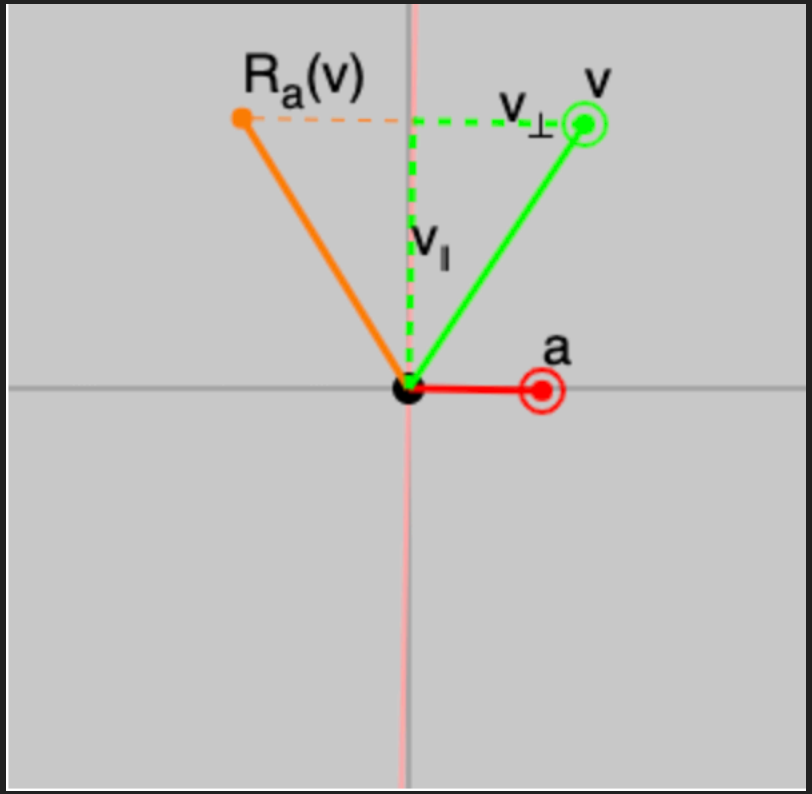


Reflection

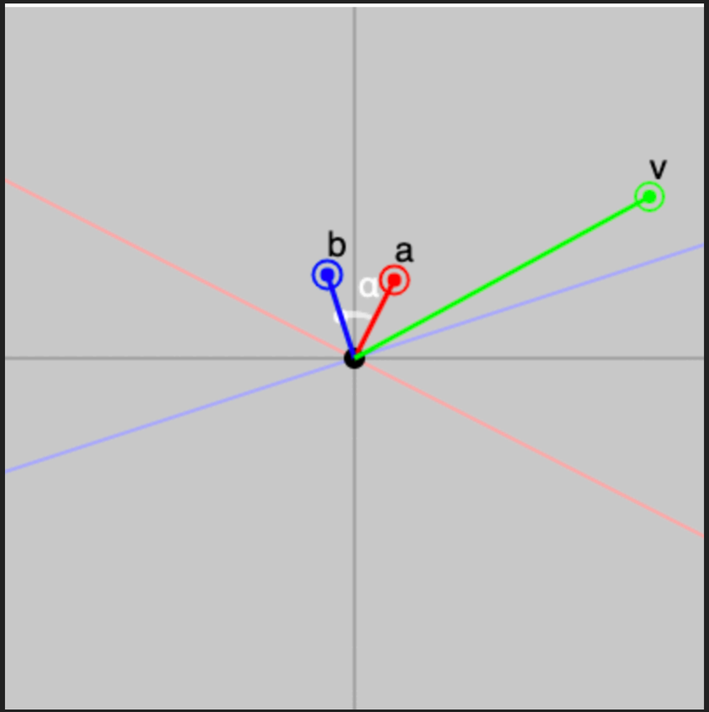
- Reflect v on plane perpendicular to a
- $v = v_\parallel + v_\perp$
- $R_a(v) = v_\parallel - v_\perp$

$$a \cdot v = \frac{1}{2}(av + va)$$

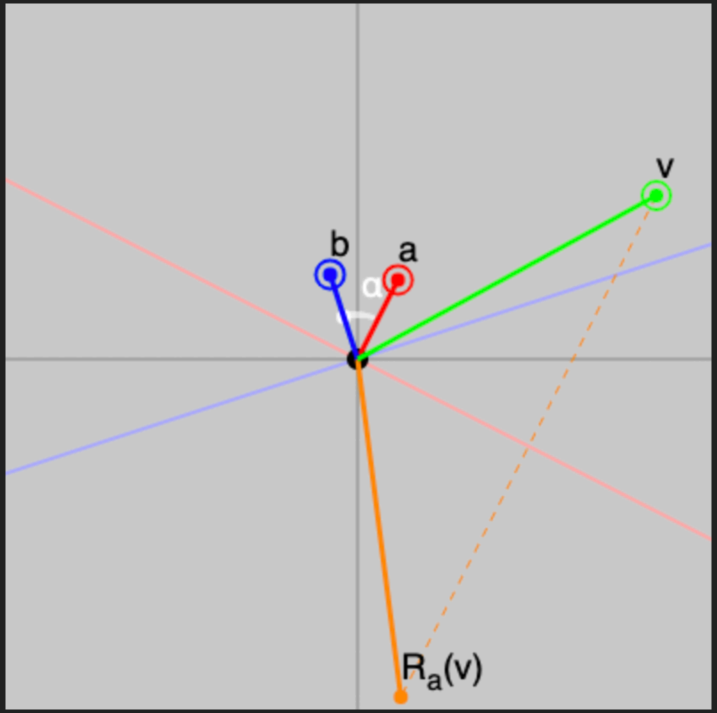
$$\begin{aligned} R_a(v) &= v_\parallel - v_\perp \\ &= v - (a \cdot v)a - (a \cdot v)a \\ &= v - 2(a \cdot v)a \\ &= v - 2\frac{1}{2}(av + va)a = v - ava - va^2 = v - ava - v \\ &= -ava \end{aligned}$$



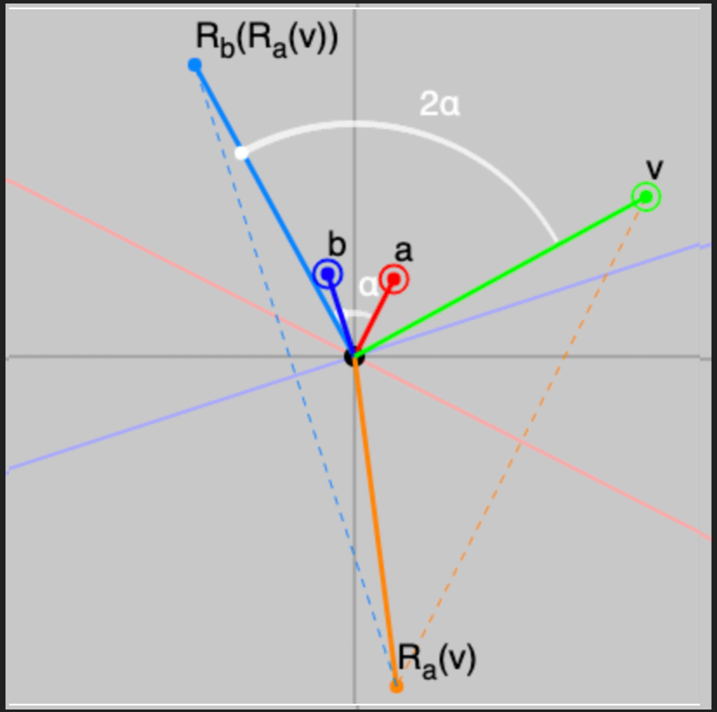
Rotation via Double Reflection



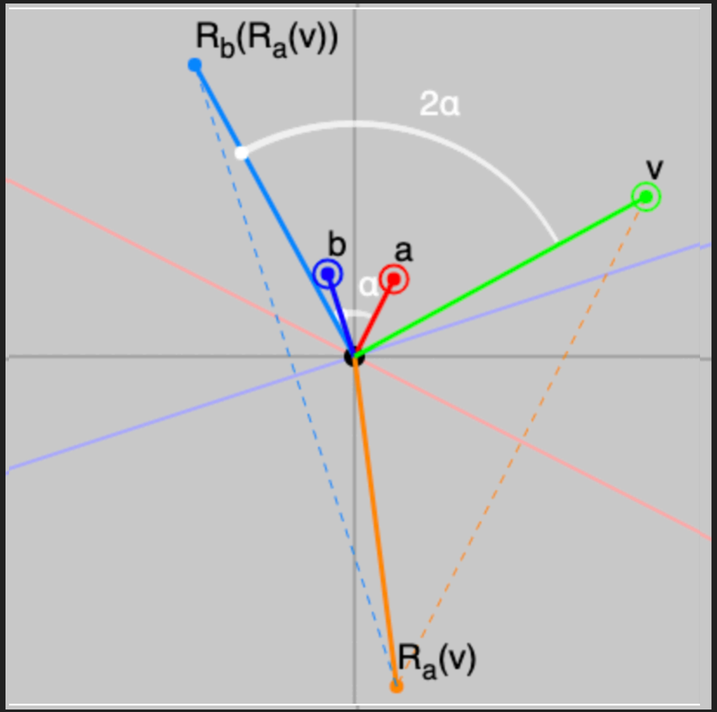
Rotation via Double Reflection



Rotation via Double Reflection

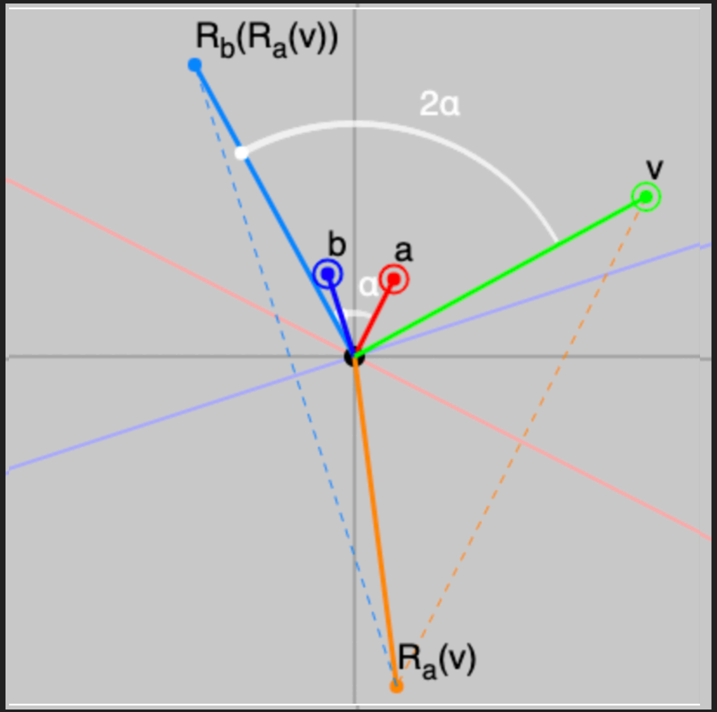


Rotation via Double Reflection



$$Rot_{2\alpha}(v) = R_b\big(R_a(v)\big) = R_b(-ava) = -b(-ava)b = bavab$$

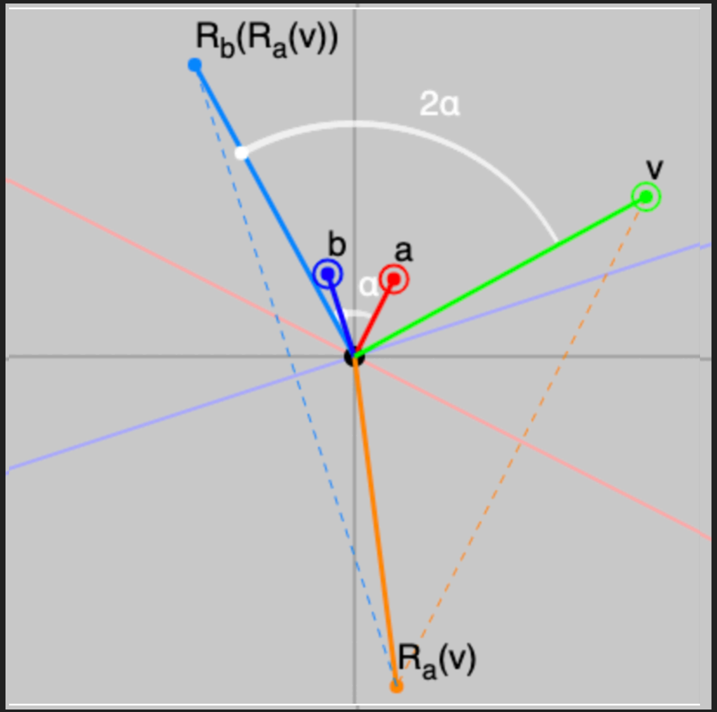
Rotation via Double Reflection



$$Rot_{2\alpha}(v) = R_b\big(R_a(v)\big) = R_b(-ava) = -b(-ava)b = bavab$$

$$(ba)(ab) = b(aa)b = bb = 1, ba = (ab)^{-1}$$

Rotation via Double Reflection



$$Rot_{2\alpha}(v) = R_b\big(R_a(v)\big) = R_b(-ava) = -b(-ava)b = bavab$$

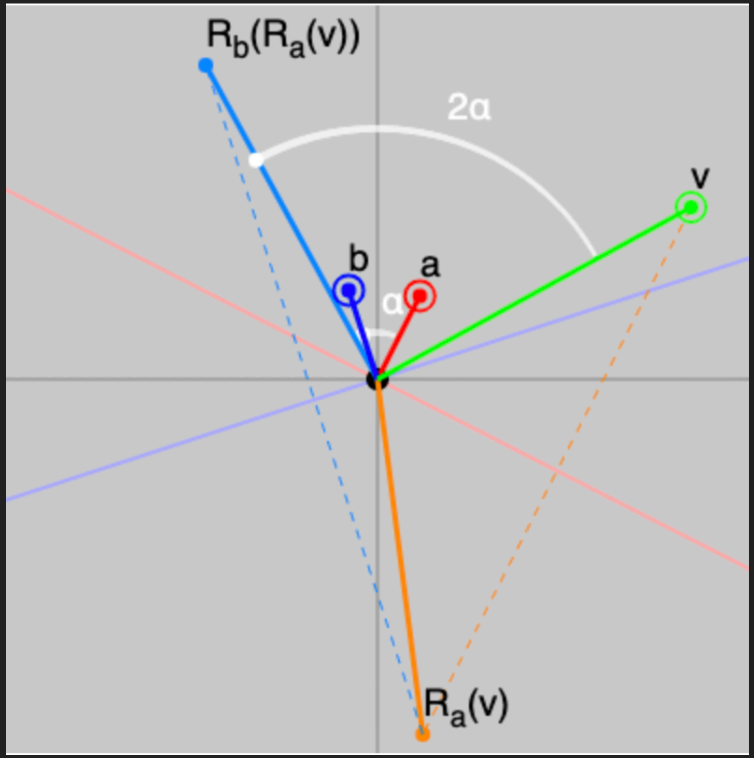
$$(ba)(ab) = b(aa)b = bb = 1, ba = (ab)^{-1}$$

$$Rot_{2\alpha}(v) = (ab)^{-1}vab$$

Rotors

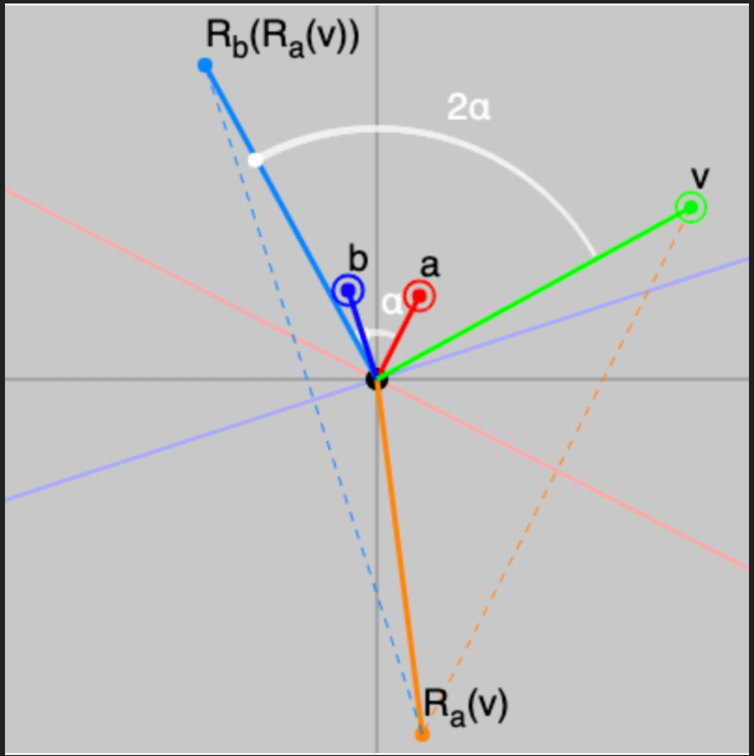
$$R = uv = u \cdot v + u \wedge v = |u||v|e^{\theta B}$$

Rotors



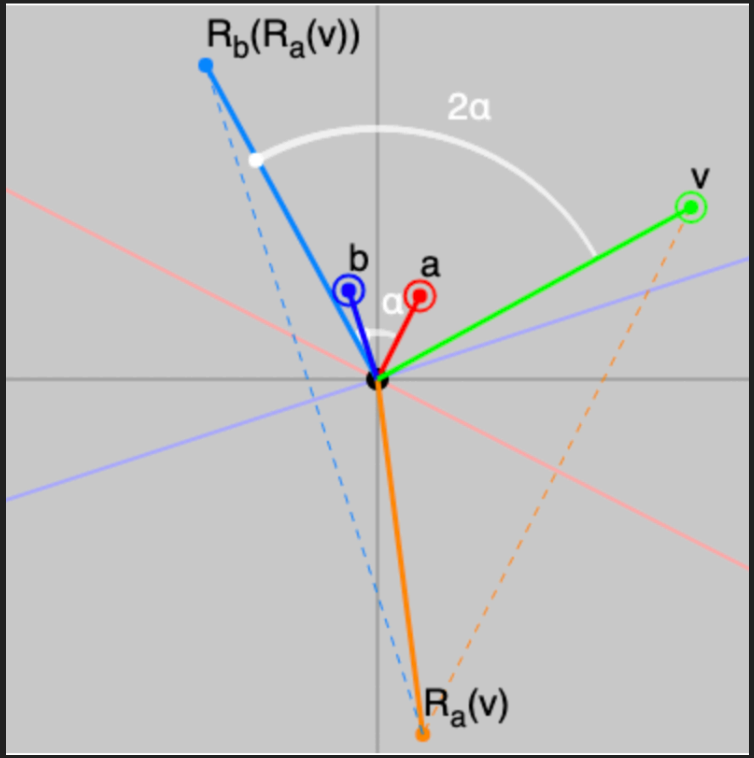
$$R = uv = u \cdot v + u \wedge v = |u||v|e^{\theta B}$$

Rotors



$$R = uv = u \cdot v + u \wedge v = |u||v|e^{\theta B}$$
$$ab = |a||b|e^{\theta B} = e^{\alpha B}$$

Rotors

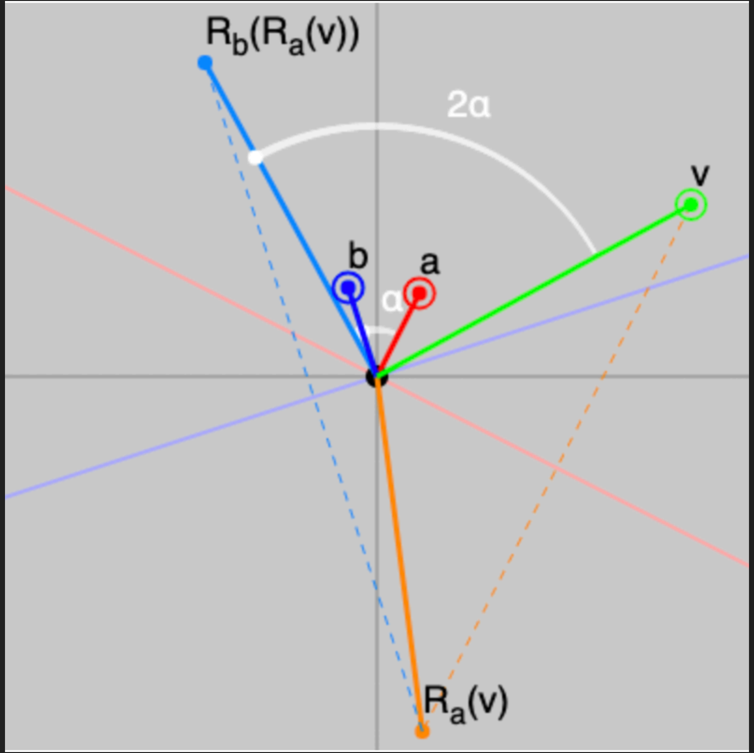


$$R = uv = u \cdot v + u \wedge v = |u||v|e^{\theta B}$$

$$ab = |a||b|e^{\theta B} = e^{\alpha B}$$

$$Rot_{2\alpha}(v) = (ab)^{-1}vab = e^{-\alpha B}ve^{\alpha B}$$

Rotors



$$R = uv = u \cdot v + u \wedge v = |u||v|e^{\theta B}$$

$$ab = |a||b|e^{\theta B} = e^{\alpha B}$$

$$Rot_{2\alpha}(v) = (ab)^{-1}vab = e^{-\alpha B}ve^{\alpha B}$$

$$Rot_{\theta,B}(v) = e^{-\frac{\theta}{2}B}ve^{\frac{\theta}{2}B}$$

Relating back

$$R = uv = u \cdot v + u \wedge v$$

$$= w + xe_1 \wedge e_2 + ye_2 \wedge e_3 + ze_1 \wedge e_3$$

Relating back

$$R = uv = u \cdot v + u \wedge v$$
$$= w + xe_1 \wedge e_2 + ye_2 \wedge e_3 + ze_1 \wedge e_3$$

$$q = w + xi + yj + zk$$

Relating back

$$R = uv = u \cdot v + u \wedge v$$

$$= w + xe_1 \wedge e_2 + ye_2 \wedge e_3 + ze_1 \wedge e_3$$

$$q = w + xi + yj + zk$$

$$e_1 \wedge e_2 = i \qquad e_2 \wedge e_3 = j \qquad e_1 \wedge e_3 = k$$

Relating back

$$R = uv = u \cdot v + u \wedge v$$

$$= w + xe_1 \wedge e_2 + ye_2 \wedge e_3 + ze_1 \wedge e_3$$

$$q = w + xi + yj + zk$$

$$e_1 \wedge e_2 = i \qquad e_2 \wedge e_3 = j \qquad e_1 \wedge e_3 = k$$

$$Quaternions \ \subset Geometric\ Algebra$$

Further Applications

Linear Algebra – System of Equations

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Linear Algebra – System of Equations

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \longrightarrow \alpha x + \beta y = b$$

Linear Algebra – System of Equations

$$\alpha x + \beta y = b$$

Linear Algebra – System of Equations

$$\alpha x + \beta y = b$$

$$\alpha x \wedge y + \beta y \wedge y = b \wedge y$$

Linear Algebra – System of Equations

$$\alpha x + \beta y = b$$

$$\alpha x \wedge y + \beta y \wedge y = b \wedge y$$

$$y \wedge y = 0$$
$$\alpha x \wedge y = b \wedge y$$

Linear Algebra – System of Equations

$$\alpha x + \beta y = b$$

$$\alpha x \wedge y + \beta y \wedge y = b \wedge y$$

$$y \wedge y = 0$$

$$\alpha x \wedge y = b \wedge y$$

$$\alpha = \frac{b \wedge y}{x \wedge y}$$

Linear Algebra – System of Equations

$$\alpha x + \beta y = b$$

$$\alpha x \wedge y + \beta y \wedge y = b \wedge y$$

$$y \wedge y = 0$$

$$\alpha x \wedge y = b \wedge y$$

$$\alpha = \frac{b \wedge y}{x \wedge y} \qquad \beta = \frac{b \wedge x}{y \wedge x} = \frac{x \wedge b}{x \wedge y}$$

Linear Algebra – System of Equations

$$|x \wedge y| = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$$

Linear Algebra – System of Equations

$$|x \wedge y| = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$$

$$\alpha = \frac{b \wedge y}{x \wedge y} = \frac{\begin{vmatrix} b_1 & y_1 \\ b_2 & y_2 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}}$$

Other Use Cases

- Geometric Calculus
  - Calculus in G^n

Other Use Cases

- Geometric Calculus
  - Calculus in G^n
- Homogeneous Geometric Algebra
  - Represent objects not centered at the origin
  - Projective Geometry

Other Use Cases

- Geometric Calculus
  - Calculus in G^n
- Homogeneous Geometric Algebra
  - Represent objects not centered at the origin
  - Projective Geometry
- Conformal Geometric Algebra
  - Better representations of points, lines, spheres, etc.
  - Generalized operations for transformations

International Workshop on Data-Mining and Statistical Science (DMSS2008)

Feature Extraction with Geometric Algebra for Semi-Supervised Learning of Time-Series Spatial Vector

Minh Tuan Pham — School of Engineering, Nagoya University
minhtuan@cmplx.cse.nagoya-u.ac.jp

Kanta Tachibana — School of Engineering, Nagoya University
kanta@fcs.coe.nagoya-u.ac.jp

Tomohiro Yoshikawa — School of Engineering, Nagoya University
yoshikawa@cse.nagoya-u.ac.jp

Takeshi Furuhashi — School of Engineering, Nagoya University
furuhashi@cse.nagoya-u.ac.jp

keywords: Geometric Algebra, Feature Extraction, Hidden Markov Model, Semi-supervised Learning

Summary

In fields of machine learning of patterns most conventional methods of feature extraction do not pay much attention to the geometric properties of data, even in cases where the data have spatial features. In this study we introduce geometric algebras to systematically extract invariant geometric features from spatial data given in a vector space. A geometric algebra is a multidimensional generalization of complex numbers and of quaternions, and able to accurately describe oriented spatial objects and relations between them. We further propose a kernel to measure similarity between two series of spatial vectors based on Hidden Markov Models. As an apllication, we demonstrate our new method with the semi-supervised learning of online hand-written digits. The result shows that the feature extraction with geometric algebra improved recognition rate in one-to-one semi-supervised learning problems of online hand-written digits.