Control Theory: Feedback and the Pole-Shifting Theorem

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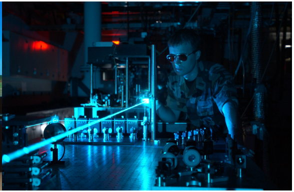
- introduce a control, u

Examples

- cruise control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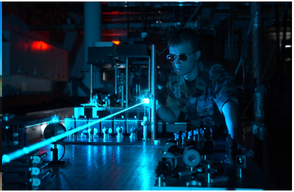


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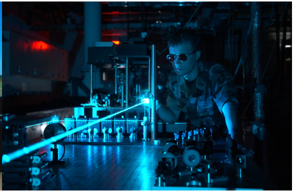
- biological motor control systems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- allows us to write $\dot{x} = \phi(t, x, u)$

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- $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$

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- $\dot{\mathbf{x}} := \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ \frac{g \sin x_1}{L} - \frac{u}{mL^2} \end{pmatrix} =: \mathbf{f}(\mathbf{x}, \mathbf{u})$

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- state vs. output feedback

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Linearity and Time-Invariance

- linear: $\dot{\mathbf{x}} = A(t)\mathbf{x} + B(t)\mathbf{u}$
- time-invariant: $\dot{\mathbf{x}} = \phi(t, \mathbf{x}, \mathbf{u}) = f(\mathbf{x}, \mathbf{u})$
- much simpler mathematically

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- $B = \begin{pmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{pmatrix}$

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- $AB = \begin{pmatrix} -\frac{1}{mL^2} \\ 0 \end{pmatrix}$

Pole-Shifting Theorem Statement

- $A(n \times n)$ and $B(n \times m)$ are s.t. $\mathit{rank}\left(\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}\right) = \mathrm{n}$

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- $\exists F(m \times n)$ s.t. eigenvalues of $A + BF$ are arbitrary

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- control system reduces to a dynamical system!

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- $$(f_2 > 0)\&(f_1 > \frac{{f_2}^2}{4mL^2} + mgL)$$
$$\rightarrow$$
$$Re(\lambda_\pm) < 0$$

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- Thanks for your time!