Understanding the Prime Number Theorem
Misunderstood Monster or Beautiful Theorem?

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1. Introduction

2. Complex Plane

3. Complex functions and Analytic Continuation

4. Gamma Function

5. Laplace Transform

6. Zeta Function

7. The Prime Number Theorem!
The Prime Number Theorem (PNT)

- Describes asymptotic behavior of $\pi(x)$
- Formally, $\pi(x) \sim \frac{x}{\log(x)}$ as $x \to \infty$

Goal

- Introduce preliminary topics necessary for the PNT
- Understand properties of functions necessary for PNT
- Briefly sketch proof of the PNT
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A complex number is a number of the form $z = x + iy$ where $z$ has both a real and imaginary component.

Each complex number is an element in the complex plane (There is a one to one correspondence between $\mathbb{C}$ and $\mathbb{R}^2$.)

We can also talk about the extended complex plane $\mathbb{C} \cup \infty$. 
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Functions exist in $\mathbb{C}$ just like in normal Euclidean n space. We can talk about differentiating and integrating these functions. (Cauchy Integral formula seen below)

$$f(a) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z - a} \, dz$$  \hspace{1cm} (1)

We can also talk about something called analytic continuation. This means extending an analytic function from its normal domain of definition.
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The Gamma function $\Gamma(z)$ extends the factorial function to the complex plane

**Gamma function**

For $Re(z) > 0$, we have:

$$\Gamma(z) = \int_{0}^{\infty} e^{-t} t^{z-1} dt \quad (2)$$

The identity $\Gamma(z + 1) = z\Gamma(z)$ arises from integration by parts. Using this identity, we can meromorphically extend $\Gamma(z)$ to the rest of $\mathbb{C}$. 
Note: We can also express the Gamma function as an infinite product. Letting $\gamma$ denote Euler’s Constant, we have:

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{k=1}^{\infty} (1 + \frac{z}{k})e^{-\frac{z}{k}}$$ (3)
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Laplace Transform

For a piecewise continuous function, $h(s)$, the Laplace transform is defined as:

$$(Lh)(z) = \int_0^\infty e^{-sz} h(s) \, ds \quad (4)$$

Aside

Interesting result: We can then write the derivative

$$\frac{d}{dz} \frac{\Gamma(z)}{\Gamma(z)} = \int_0^\infty e^{-sz} g(s) \, ds$$

Where $g(s) = \frac{s}{1-e^{-s}}$

Finally, we get an asymptotic relationship for Gamma:

$$\Gamma(z) = z^z e^{-z} \sqrt{\frac{2\pi}{z}} \left(1 + \frac{1}{12z} + O\left(\frac{1}{n^2}\right)\right)$$
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The Zeta function (Euler) is represented by

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \text{ for } \Re(s) > 1 \]  

(5)

We can see the more explicit connection of \( \zeta(s) \) and the primes if we look at the infinite product representation of the zeta function:

\[ \zeta(s) = \prod_p \frac{1}{1-p^{-s}} \text{ for } \Re(s) > 1 \]  

(6)
Now we want to extend Zeta to the entire complex plane. How? A branch cut here... an Integral there... and a lot of magic. It turns out that the Zeta function can be meromorphically extended to the complex plane. It has one simple pole at $s = 1$.

More formally, it satisfies the equation

$$
\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1 - s) \zeta(1 - s)
$$

(7)
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Stated again, formally: The number of primes, $\pi(x)$, not bigger than $x$ satisfies

$$\pi(x) \sim \frac{x}{\log(x)} \text{ as } x \to \infty$$  \hspace{1cm} (8)

The proof of the PNT is pretty messy (and magical according to Dr. Gamelin), but it relies heavily upon the following functions:

$$\Phi(s) = \sum_p \frac{\log(p)}{p^s} (Re(s) > 1) \hspace{1cm} (9)$$

$$\theta(x) = \sum_{p \leq x} \log(p) \hspace{1cm} (10)$$
First, the proof involves showing that $\zeta(s)$ does not have any zeros on the line $Re(s) = 1$. Essentially, the rest of the proof boils down to proving that $\theta(x) \sim x$, but to do that, we look at the Laplace transform of a nasty variation of $\theta(x)$ and a tricky contour integral ... and tada! we have that $\frac{\theta(x)}{x} \sim 1$, and by squeeze, we have the PNT.

Interesting identity: $\pi(x) \sim \int_2^x \frac{1}{\log(t)} dt$
A special thanks to the entire DRP program for this opportunity. Especially to Nathaniel Monson for putting up with my questions.
Resources

If you want to improve this style

- Nathaniel Monson’s brain
- Complex Analysis, T. Gamelin