Understanding the Prime Number Theorem

Misunderstood Monster or Beautiful Theorem?

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Outline

1 Introduction

2 Complex Plane

3 Complex functions and Analytic Continuation

4 Gamma Function

5 Laplace Transform

6 Zeta Function

7 The Prime Number Theorem!

Outline and Goals

The Prime Number Theorem (PNT)

- Describes asymptotic behavior of $\pi(x)$
- Formally, $\pi(x) \sim \frac{x}{log(x)}$ as $x \to \infty$

Goal

- Introduce preliminary topics necessary for the PNT
- Understand properties of functions necessary for PNT
- Briefly sketch proof of the PNT

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Complex Plane

- A complex number is a number of the form $z = x + iy$ where z has both a real and imaginary component.
- Each complex number is an element in the complex plane (There is a one to one correspondance between $\mathbb{C}$ and $\mathbb{R}^2$.)
- We can also talk about the extended complex plane $\mathbb{C} \cup \infty$.

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Complex functions and Analytic Continuation

Functions exist in $\mathbb{C}$ just like in normal Euclidean n space. We can talk about differentiating and integrating these functions. (Cauchy Integral formula seen below)

$$f(a) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z-a}\, dz \tag{1}$$

We can also talk about something called analytic continuation. This means extending an analytic function from its normal domain of definition.

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Gamma Function

The Gamma function $\Gamma(z)$ extends the factorial function to the complex plane

Gamma function

For $Re(z) > 0$, we have:

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \tag{2}$$

The identity $\Gamma(z+1) = z\Gamma(z)$ arises from integration by parts. Using this identity, we can meromorphically extend $\Gamma(z)$ to the rest of $\mathbb{C}$.

Gamma Function contd

Note: We can also express the Gamma function as an infinite product. Letting γ denote Euler's Constant, we have:

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{k=1}^{\infty}(1+\frac{z}{k})e^{-\frac{z}{k}} \tag{3}$$

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Laplace Transform

For a piecewise continuous function, $h(s)$, the Laplace transform is defined as:

$$(\mathrm{Lh})(\mathrm{z}) = \int_0^\infty e^{-sz} h(s) ds \tag{4}$$

Aside

Interesting result: We can then write the derivative $\frac{d}{dz}\frac{\Gamma\prime(z)}{\Gamma(z)} = \int_0^\infty e^{-sz} g(s) ds$ Where $g(s) = \frac{s}{1-e^{-s}}$
Finally, we get an asymptotic relationship for Gamma:
$\Gamma(z) = z^z e^{-z}\sqrt{\frac{2\pi}{z}}(1 + \frac{1}{12z} + \mathcal{O}(\frac{1}{n^2}))$

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The Zeta Function

The Zeta function (Euler) is represented by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \text{ for } Re(s) > 1 \tag{5}$$

We can see the more explicit connection of $\zeta(s)$ and the primes if we look at the infinite product representation of the zeta function:

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}} \text{ for } Re(s) > 1 \tag{6}$$

Zeta Function contd

Now we want to extend Zeta to the entire complex plane. How? A branch cut here... an Integral there... and a lot of magic. It turns out that the Zeta function can be meromorphically extended to the complex plane. It has one simple pole at s = 1.
More formally, it satisfies the equation

$$\zeta(s) = 2^s \pi^{s-1} sin(\frac{\pi s}{2}) \Gamma(1-s) \zeta(1-s) \tag{7}$$

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PNT

Stated again, formally: The number of primes, $\pi(x)$, not bigger than x satisfies

$$\pi(x) \sim \frac{x}{log(x)} \text{ as } x \to \infty \tag{8}$$

The proof of the PNT is pretty messy (and magical according to Dr. Gamelin), but it relies heavily upon the following functions:

$$\Phi(s) = \sum_p \frac{log(p)}{p^s}(Re(s) > 1) \tag{9}$$

$$\theta(x) = \sum_{p \leq x} log(p) \tag{10}$$

PNT contd

First, the proof involves showing that $\zeta(s)$ does not have any zeros on the line $Re(s) = 1$. Essentially, the rest of the proof boils down to proving that $\theta(x) \sim x$, but to do that, we look at the Laplace transform of a nasty variation of $\theta(x)$ and a tricky contour integral ... and tada! we have that $\frac{\theta(x)}{x} \sim 1$, and by squeeze, we have the PNT.

Interesting identity: $\pi(x) \sim \int_2^x \frac{1}{log(t)} dt$

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Resources

If you want to improve this style

- Nathaniel Monson's brain
- Complex Analysis, T. Gamelin