First-Order Logic++

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Basics

Here are some of the basic things

- Languages
- Sentence
- L-structures/Models
  - The Language of Groups: $(\cdot, ^{-1}, e)$ has the structure $\mathbb{R} \setminus \{0\}$
  - The Language of Rings: $(\cdot, +, 0, 1)$ has the structure $\mathbb{Z}$
- The symbol $\models$
  - $\mathscr{A} \models \sigma$: the formula σ is true in the model $\mathscr{A}$.
  - $\Gamma \models \sigma$: every L-structure $\mathscr{A}$ that models Γ also models σ

Proofs

The symbol $\vdash$

- $\Gamma \vdash \sigma$: there exists a proof from Γ to σ

But what is a proof?

- A finite sequence of sentences where each sentence is something from your Proof System.

Proof System

- Λ: Logical Axioms
  - $\forall x P(x) \rightarrow \exists x P(x)$
  - $A \vee \neg A$
- Γ: Assumptions/Axioms
  - $\forall x, y(x \cdot y = y \cdot x)$
  - $\underbrace{1 + 1 + ... + 1}_{p} = 0$
- Results derived from Modus Ponens ($\alpha \rightarrow \beta$, α, so $\therefore \beta$)

Soundness and Completeness

Soundness

- $\Gamma \vdash \sigma \Rightarrow \Gamma \models \sigma$

Completeness

- $\Gamma \models \sigma \Rightarrow \Gamma \vdash \sigma$

We say Γ is satisfiable if there exists a structure $\mathscr{A}$ such that $\mathscr{A} \models \Gamma$
We say Γ is consistent if $\Gamma \nvdash \bot$, i.e. that there is no proof of contradiction.

Additionally:

- Soundess $\Leftrightarrow$ (Γ Satisfiable $\Rightarrow$ Γ Consistent)
- Completeness $\Leftrightarrow$ (Γ Consistent $\Rightarrow$ Γ Satisfiable)
- Hence: Γ Satisfiable $\Leftrightarrow$ Γ Consistent

Statement of the Theorem

Theorem: For all fields $\mathscr{F}$ that model ACF_p or ACF_0, if $f : F^n \to F^n$ is an injective polynomial function, then it must also be surjective

Corollary: If $f : \mathbb{C}^n \to \mathbb{C}^n$ is an injective polynomial function, then it must also be surjective.

Some Preliminary details

Our Proof System

- Λ: Logical Axioms.
- Γ: Field Axioms.
- $\alpha_p : \underbrace{1+1+1+...+1}_{p} = 0$ for p prime
  - Fact: $\Gamma \cup \{\alpha_p\} \vdash \neg\alpha_q$ for all primes $q \neq p$
- $\psi_n : \forall a_0 \forall a_1 ... \forall a_n \exists x (a_0 + a_1 x + ... + a_n x^n = 0)$
- ACF_p: $\Gamma \cup \alpha_p \cup \{\psi_n\}_{n \in \mathbb{N}}$.
- ACF_0: $\Gamma \cup \{\neg\alpha_p\}_{p\ prime} \cup \{\psi_n\}_{n \in \mathbb{N}}$.

Fact: ACF_p and ACF_0 are complete theories.

- $T \cup \{\sigma\}$ is satisfiable $\Rightarrow \sigma \in T$ or
- $T \models \sigma \vee T \models \neg\sigma$

More Preliminary details

Consider the field:

$$\mathbb{F}_p := \frac{\mathbb{Z}}{p\mathbb{Z}}$$

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This is the field we will be working with for this proof.

A Final Observation

For any two fields

$$\mathbb{F}_{p^r},\ \mathbb{F}_{p^s}$$

there is always a field above both of them, e.g.

$$\mathbb{F}_{p^{lcm(r,s)}}$$

Some Algebra

Easy to show that F has characteristic p

- 1 is still 1, so $\underbrace{1 + 1 + ... + 1}_{p}$ is still 0.

Lemma: F is algebraically closed (and hence $F \models ACF_p$)
Proof: If you know algebra, easy to show, but will not prove here.

Some Simpler Algebra

I now wish to prove the Ax-Grothendieck Theorem for $F = \bigcup_{k\in\mathbb{N}} \mathbb{F}_{p^k}$. Let $\phi_{n,d}$ be the formula stating that all n-tuples of polynomials of at most degree d which are injective (as functions $F^n \to F^n$) are surjective.

Proof: Let f be an injective polynomial function from F^n to F^n where each coordinate function is of at most degree d.

Let r be such that all of the coefficients of all of the coordinate functions (of which there are a finite amount) are in $\mathbb{F}_{p^r}$

Assume f is not surjective. Thus there must be some $x_0 \in F^n$ not in the image of f. Since $x_0 \in (\bigcup_{k\in\mathbb{N}} \mathbb{F}_{p^k})^n$, let s be such that $x_0 \in \mathbb{F}_{p^s}^n$.

Finally, let $m := lcm(r, s)$, which then means that all of the coefficients of f and the coordinates of x_0 are members of $\mathbb{F}_{p^m}$.

Some Simpler Algebra

Thus we have that $f : \mathbb{F}_{p^m}^n \to \mathbb{F}_{p^m}^n$ is injective but not surjective since it misses x_0.

However, since $\mathbb{F}_{p^m}^n$ is finite, f is injective, and (clearly) $\mathbb{F}_{p^m}$ is of the same size as $\mathbb{F}_{p^m}$, that must mean that f is surjective. Since we assumed that it wasn't, we get a contradiction.

Thus f is surjective and $F \models \phi_{n,d}$.

Generalization

Now that Ax-Grothendieck is true for some model F of ACF_p, I wish to show that this means it's true for all models.

Proof: We now have that $F \models ACF_p$ and $F \models \phi_{n,d}$. This is equivalent to saying $F \models ACF_p \cup \{\phi_{n,d}\}$, which by definition means that $ACF_p \cup \{\phi_{n,d}\}$ is satisfiable.

Since ACF_p is a complete theory, this means by the first definition we used that $\phi_{n,d} \in ACF_p$. Since this statement contains no mention of models, it must hold regardless of model and hence be true for all models.

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The Actual Logic

Finally, I wish to show that Ax-Grothendieck is true in ACF_0 (and hence true for $\mathbb{C}$).

Proof: Assume there is some $\phi_{n,d}$ such that $ACF_0 \not\models \phi_{n,d}$.

Since ACF_0 is a complete theory, by the second (equivalent) definition we have that $ACF_0 \models \neg\phi_{n,d}$. By completeness this means that $ACF_0 \vdash \neg\phi_{n,d}$

Since proofs are finite, that must mean that in a proof from ACF_0 to $\neg\phi_{n,d}$ there were at most a finite amount of $\neg\alpha_p$'s. Let q be a prime such that

$$q > max\{p|\neg\alpha_p \text{ appears in the proof from } ACF_0 \text{ to } \neg\phi_{n,d}\}$$

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By that fact from earlier, this means that all $\neg\alpha_p$ also hold in ACF_q.

Hence the proof from ACF_0 to $\neg\phi_{n,d}$ is also a proof from ACF_q to $\neg\phi_{n,d}$, which contradicts what we already proved earlier.

$\therefore ACF_0 \models \phi_{n,d}$ for all $\phi_{n,d}$.

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Bibliography

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- Sam Bloom

Questions?