An Application of Sard's Theorem to Electrostatics

Directed Reading Program: Differential Topology

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Some Basic Definitions

- **Smooth Mapping:** A mapping from an open set is smooth if it has continuous partial derivatives of all orders (class C^∞). In general, any map is smooth if it can be locally extended to a smooth map on open sets.
- **Manifold:** A subset X of $\mathbb{R}^n$ for which there exists a smooth map between X and $\mathbb{R}^k$, which is bijective and whose inverse is also smooth (this type of function is a diffeomorphism).
- **Tangent Space:** The image of the derivative mapping $d\phi_x : \mathbb{R}^k \to \mathbb{R}^n$, where ϕ_x is some parameterization of a k dimensional manifold about x.
- **Immersion:** A function $f : X \to Y$ is an immersion at a point if the derivative mapping $df_x : T_x(X) \to T_{f(x)}(Y)$ is injective at that point. If the function is an immersion at every point, it is an immersion.
- **Submersion:** A function f is a submersion at a point if the derivative mapping is surjective at that point. If the function is a submersion at every point, it is a submersion.

Regular and Critical Values and Points

The following are defined for a smooth map on manifolds $f : X \to Y$.

- **Regular Point:** A regular point is a point in X at which the derivative mapping is surjective (f is a submersion at that point).
- **Critical Point:** A point is a critical point if it is not a regular point.
- **Regular Value:** A point in $y \in Y$ is a regular value if the derivative mapping is surjective for all $x \in X$ such that $f(x) = y$.
- **Critical Value:** A point $y \in Y$ is a critical value if it is not a regular value, that is, if there is at least one x such that $f(x) = y$ and the derivative mapping is not a submersion at that point.
- **Non-Degenerate Critical Point:** A critical point of a function $f : X \to \mathbb{R}$ is non-degenerate if the Hessian matrix is nonsingular at the critical point.
- **Morse Function:** A function whose critical points are all non-degenerate.

Lebesgue Measure Zero

A set has Lebesgue measure zero if the set can be covered by a countably infinite number of rectangular n-solids in $\mathbb{R}^n$ with an arbitrarily small total volume. That is to say, for all $\varepsilon > 0$, there is a countably infinite collection of solids $(S_1, S_2, \dots)$ such that the collection covers the set and

$$\sum_{i=1}^{\infty} \mathrm{vol}(S_i) < \varepsilon$$

A measure zero set is "small" compared to a set with measure greater than zero. When the statement *almost every* is used in a logical statement, it means the set of values for which the statement is false has measure zero.

Sard's Theorem

These are two equivalent statements of Sard's Theorem:

Theorem

The set of critical values of a smooth map of manifolds $f : X \to Y$ has measure zero.

Theorem

If $f : X \to Y$ is a smooth map of manifolds, then almost every point in Y is a regular value of f.

Why is Sard's Theorem Important?

One theorem which is very useful is the Preimage Theorem:

Theorem

If y is a regular value of $f : X \to Y$, then the preimage $f^{-1} = (y)$ is a submanifold of X.

Combining this with Sard's Theorem, we know that since almost all values of a mapping are regular, the preimage of a mapping is almost always a manifold. There are many other interesting and useful results which rely on Sard's Theorem (The Whitney Immersion Theorem, The Whitney Embedding Theorem, etc).

A Consequence in Morse Theory

There is one additional result which is very useful which is proven using Sard's Theorem:

Lemma

For any immersion $\phi : X \to \mathbb{R}^n$ with coordinate functions $\phi_1, \phi_2, \ldots, \phi_n$, for almost every $a_1, a_2, \ldots, a_n$ the function $a_1\phi_1 + a_2\phi_2 + \cdots + a_n\phi_n$ is a Morse function on X.

An Application to Electrostatics

Consider the following system: four point charges sitting in $\mathbb{R}^3$ such that all four point charges are not in a plane. Using the previous consequence of Sard's Theorem, we can show the following:

Proposition

For any four point charges with charge values q_i located at $\mathbf{p_i}$ in $\mathbb{R}^3$ such that all four charges do not lie in a plane, for almost every choice of $\mathbf{q} = \{q_1, q_2, q_3, q_4\}$ the equilibrium (critical) points of the potential function

$$V_{\mathbf{q}} = \frac{q_1}{r_1} + \frac{q_2}{r_2} + \frac{q_3}{r_3} + \frac{q_4}{r_4}$$

(where $r_i = |\mathbf{x} - \mathbf{p_i}|$) are non-degenerate.

An Application to Electrostatics

First, we consider the smooth map $\psi : \mathbb{R}^3 \setminus \{x_1, x_2, x_3, x_4\} \to \mathbb{R}^4$, with coordinate functions r_1, r_2, r_3, r_4. We look at the Jacobian of this function at any given point $\mathbf{x} = (x, y, z)$:

$$J(\mathbf{x}) = \begin{pmatrix} \frac{x-p_{1x}}{r_1} & \frac{y-p_{1y}}{r_1} & \frac{z-p_{1z}}{r_1} \\ \frac{x-p_{2x}}{r_2} & \frac{y-p_{2y}}{r_2} & \frac{z-p_{2z}}{r_2} \\ \frac{x-p_{3x}}{r_3} & \frac{y-p_{3y}}{r_3} & \frac{z-p_{3z}}{r_3} \\ \frac{x-p_{4x}}{r_4} & \frac{y-p_{4y}}{r_4} & \frac{z-p_{4z}}{r_4} \end{pmatrix}$$

An Application to Electrostatics

We want to look at the kernel of this matrix at any given point, which is given by all (a, b, c) such that:

$$\begin{pmatrix} \frac{x-p_{1x}}{r_1} & \frac{y-p_{1y}}{r_1} & \frac{z-p_{1z}}{r_1} \\ \frac{x-p_{2x}}{r_2} & \frac{y-p_{2y}}{r_2} & \frac{z-p_{2z}}{r_2} \\ \frac{x-p_{3x}}{r_3} & \frac{y-p_{3y}}{r_3} & \frac{z-p_{3z}}{r_3} \\ \frac{x-p_{4x}}{r_4} & \frac{y-p_{4y}}{r_4} & \frac{z-p_{4z}}{r_4} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Looking at the system of linear equations which gives us the kernel, we can see that it is equivalent to taking the inner product of a point in $\mathbb{R}^3$ with each row of the Jacobian. The four rows are a spanning set for $\mathbb{R}^3$ (since all four points do not lie on the same plane), so the only way for the matrix product to be zero is for the kernel to consist of only the zero vector. Thus, the matrix is injective and therefore our function ψ is an immersion.

An Application to Electrostatics

Define W to be the set of all points in $\mathbb{R}^4$ such that none of its coordinates are zero. Now consider the smooth mapping $\phi : W \rightarrow \mathbb{R}^4$ which takes each entry and inverts it: $\phi : (a, b, c, d) \rightarrow (\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d})$. The Jacobian of this mapping is a diagonal matrix with all of its diagonal entries non-zero (due to our choice of W), which is injective. This shows us that this function is an immersion as well. Now using the fact that compositions of immersions remain immersions, we can see that when we compose the two functions we have defined:

$$\psi : (x, y, z) \rightarrow (r_1, r_2, r_3, r_4)$$

$$\phi : (a, b, c, d) \rightarrow \left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}\right)$$

our resulting function is still an immersion.

An Application to Electrostatics

This shows us that the function $\xi = \phi \circ \psi$ is an immersion which takes in a point in $\mathbb{R}^3$ and sends it to $(\frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}, \frac{1}{r_4})$ Now we can apply the lemma from Morse Theory:

Lemma

For any immersion $\phi : X \to \mathbb{R}^n$ with coordinate functions $\phi_1, \phi_2, \dots, \phi_n$, for almost every $a_1, a_2, \dots, a_n$ the function $a_1\phi_1 + a_2\phi_2 + \cdots + a_n\phi_n$ is a Morse function on X.

The potential function $V_{\mathbf{q}} = \xi(\mathbf{x}) \cdot \mathbf{q}$ is therefore a Morse function for almost every $\mathbf{q}$. This means the critical points of $V_{\mathbf{q}}$ are almost always nondegenerate. This implies that the critical points of this function are all isolated.

The Meaning of Critical Points

From Electrostatics, we know that

$$-\nabla V = \mathbf{E}$$

For any scalar potential which maps a point in $\mathbb{R}^3$ to a point in $\mathbb{R}$, we can see that the gradient and the Jacobian are the same 1 by 3 matrix:

$$J(\mathbf{x}) = (\partial_x V, \partial_y V, \partial_z V)$$

At a critical point, we know that the Jacobian of the potential mapping is not surjective: that is

$$a\partial_x V + b\partial_y V + c\partial_z V = d$$

must fail for some $a, b, c, d \in \mathbb{R}$. Now if any of $\partial_i V$ is non-zero, say $\partial_x V$, we can choose $b = c = 0$ and chose $a\partial_x V$ to match d. Therefore, the only way for this Jacobian to not be surjective is for all three components to be identically zero. Thus it follows that $\mathbf{E} = 0$, which means that any point charge placed at the critical point will experience no net force, and thus remain stationary.