FINITE SUBGROUPS OF THE 3D ROTATION GROUP

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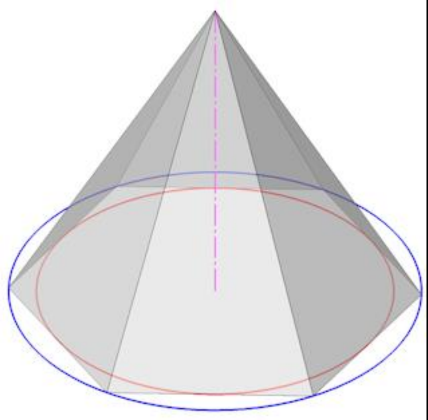
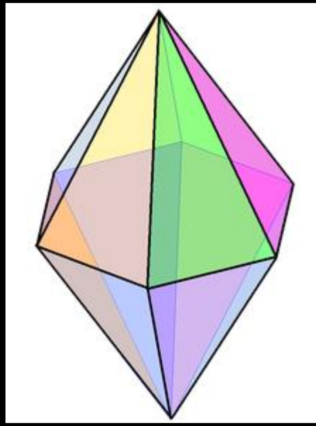
SYMMETRIES OF THE SPHERE

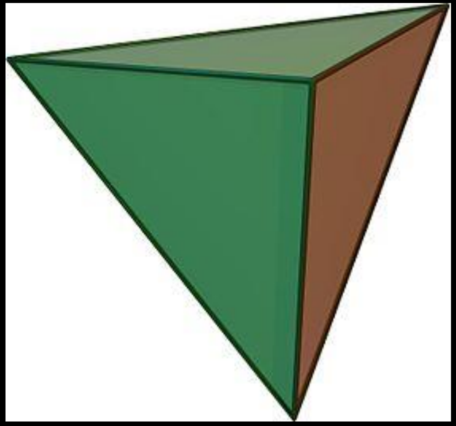
- Let’s think about the rigid rotations of the sphere.
- Rotations can be composed together to form new rotations.
- Every rotation has an inverse rotation.
- There is a “non-rotation,” or an identity rotation.
- For us, the distinct rotational symmetries of the sphere form a group, where each rotation is an element and our operation is composing our rotations.
- This group happens to be infinite, since you can continuously rotate the sphere.
- What if we wanted to study a smaller collection of symmetries, or in some sense a subgroup of our rotations?

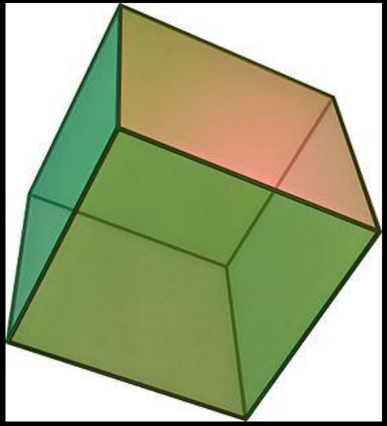
THEOREM

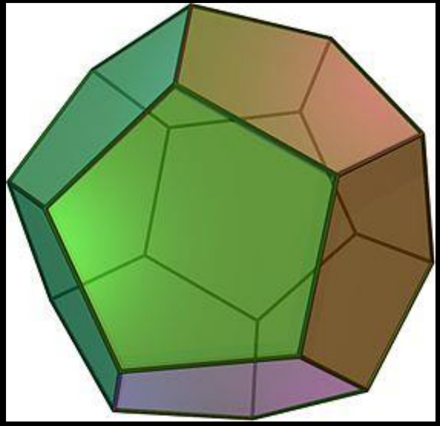
- A finite subgroup of SO_3 (the group of special rotations in 3 dimensions, or rotations in 3D space) is isomorphic to either a cyclic group, a dihedral group, or a rotational symmetry group of one of the platonic solids.
- These can be represented by the following solids :


Cyclic


Dihedral


Tetrahedron


Cube


Dodecahedron

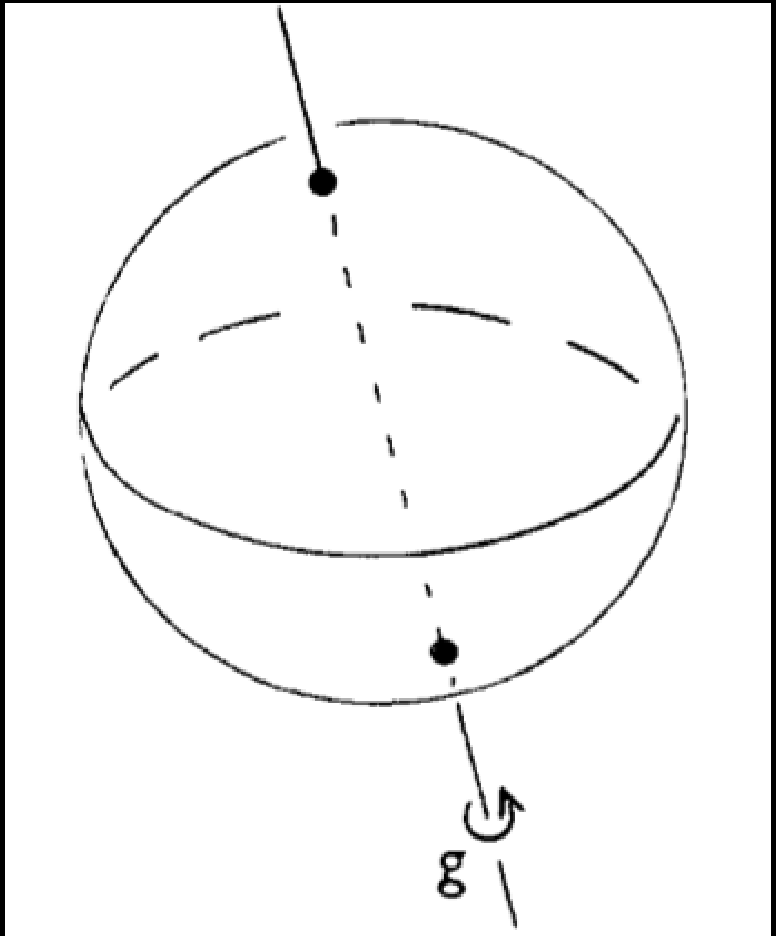
- The rotational symmetries of the cube and octahedron are the same, as are those of the dodecahedron and icosahedron.

GROUP ACTIONS

- The interesting portion of study in group theory is not the study of groups, but the study of how they act on things.
- A group action is a form of mapping, where every element of a group G represents some permutation of a set X.
  - For example, the group of permutations of the integers 1,2, 3 acting on the numbers 1, 2, 3, 4. So for example, the permutation (1,2,3) -> (3,2,1) will swap 1 and 3.
- An orbit is a collection of objects that can be permuted by the actions of the group.
  - Under our example action, the orbit of 1 is {1, 2, 3}, while the orbit of 4 is {4}.
- A stabilizer is a collection of group elements that send a given set element onto itself.
  - Under our example action, the stabilizer of 1 is only the group element {(1,2,3)->(1,2,3)}, while the stabilizer of 4 is the full group.

PROOF (SKETCH)

- Let G be a finite subgroup of SO_3. Each element of G represents a rotation of 3D space about an axis that passes through the origin, besides the identity rotation.
- We define the poles of a rotation g in G to be the two points on the unit sphere to be left fixed by g acting on 3D space.
- Let X denote the set of all poles of all elements of G, our subgroup, other than the identity element.
- We have an action of G on X.



PROOF (COUNTING ARGUMENT)

- Let N denote the number of distinct orbits, and choose a pole for each orbit. Call these x_1, x_2, … x_n. Every element of G – {e}, the identity, fixes exactly 2 poles, while the identity fixes them all.
- Here, we use the Counting Theorem :
  - The number of distinct orbits of group G acting on set X is equal to

$$\frac{1}{|G|} \sum_{g \in G} |X^g|$$

  - Where $|X^g|$ is the number of elements of X left fixed by group element G.
- For this group action, using the property above, we have :

$$N = \frac{1}{|G|} \{2(|G| - 1) + |X|\}$$

PROOF (BOUNDS ON N)

- With some algebraic manipulation of the last expression, we are left with :

$$2\left(1-\frac{1}{|G|}\right)=\sum_{i=1}^{N}\left(1-\frac{1}{|stabilizer(x_i)|}\right)$$

- Assuming G is not the trivial group {e} of size 1, the left size of the equation must be greater than or equal to 1 and less than 2, since |G| > 1.
- In addition, each stabilizer has order at least 2 (the poles), so each term of the sum on the right is greater than or equal to ½ and less than 1.
- Thus, N is either 2 or 3.

PROOF (CASES)

- If N = 2, then we have $2 = |G(x_1)| + |G(x_2)|$, and there can only be 2 poles. Every element in G must therefore rotate around the axis formed by these two poles, and the plane perpendicular to this axis is mapped onto itself. Therefore, G is isomorphic to a rotation in 2 dimensions and is a cyclic group.
- If N = 3, we have through some algebraic manipulation :

$$1 + \frac{2}{|G|} = \frac{1}{|G_x|} + \frac{1}{|G_y|} + \frac{1}{|G_z|}.$$

- Note that the terms $|G_x|$, $|G_y|$, and $|G_z|$ must be integers and that the right hand side of the equation must be greater than 1, so we have a set of possible solutions :

(a) $\frac{1}{2}, \frac{1}{2}, \frac{1}{n}$ where $n \geqslant 2$;
(b) $\frac{1}{2}, \frac{1}{3}, \frac{1}{3}$;
(c) $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$;
(d) $\frac{1}{2}, \frac{1}{3}, \frac{1}{5}$.

PROOF (MORE CASES)

- By continuing through the casework, we have the following :
  - If we are in situation (a), with 1/2, 1/2, 1/n, we have a dihedral group.
  - In situation (b), with 1/2, 1/3, 1/3, we have a regular tetrahedron.
  - In situation (c), with 1/2, 1/3, 1/4, we have the vertices of a regular octahedron, and equivalently the faces of a cube.
  - In situation (d), with 1/2, 1/3, 1/5, we have the vertices of a regular icosahedron, and equivalently the faces of a dodecahedron.
- These are therefore all of the finite subgroups of the group of rotations on 3 dimensions.

CREDIT

- Jacky Chong, mentor
- Naïve Lie Theory, by John Stillwell, reference textbook
- Groups and Symmetry, by M. A. Armstrong, reference textbook, images