FINITE SUBGROUPS OF THE 3D ROTATION GROUP

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• Let’s think about the rigid rotations of the sphere.
• Rotations can be composed together to form new rotations.
• Every rotation has an inverse rotation.
• There is a “non-rotation,” or an identity rotation.
• For us, the distinct rotational symmetries of the sphere form a group, where each rotation is an element and our operation is composing our rotations.
• This group happens to be infinite, since you can continuously rotate the sphere.
• What if we wanted to study a smaller collection of symmetries, or in some sense a subgroup of our rotations?
A finite subgroup of $SO_3$ (the group of special rotations in 3 dimensions, or rotations in 3D space) is isomorphic to either a cyclic group, a dihedral group, or a rotational symmetry group of one of the platonic solids.

These can be represented by the following solids:

- Cyclic
- Dihedral
- Tetrahedron
- Cube
- Dodecahedron

The rotational symmetries of the cube and octahedron are the same, as are those of the dodecahedron and icosahedron.
The interesting portion of study in group theory is not the study of groups, but the study of how they act on things.

A group action is a form of mapping, where every element of a group $G$ represents some permutation of a set $X$.

- For example, the group of permutations of the integers 1, 2, 3 acting on the numbers 1, 2, 3, 4. So for example, the permutation $(1,2,3) \rightarrow (3,2,1)$ will swap 1 and 3.

An orbit is a collection of objects that can be permuted by the actions of the group.
- Under our example action, the orbit of 1 is $\{1, 2, 3\}$, while the orbit of 4 is $\{4\}$.

A stabilizer is a collection of group elements that send a given set element onto itself.
- Under our example action, the stabilizer of 1 is only the group element $\{(1,2,3)\rightarrow(1,2,3)\}$, while the stabilizer of 4 is the full group.
PROOF (SKETCH)

- Let $G$ be a finite subgroup of $SO_3$. Each element of $G$ represents a rotation of 3D space about an axis that passes through the origin, besides the identity rotation.
- We define the poles of a rotation $g$ in $G$ to be the two points on the unit sphere to be left fixed by $g$ acting on 3D space.
- Let $X$ denote the set of all poles of all elements of $G$, our subgroup, other than the identity element.
- We have an action of $G$ on $X$. 
PROOF (COUNTING ARGUMENT)

• Let $N$ denote the number of distinct orbits, and choose a pole for each orbit. Call these $x_1, x_2, \ldots, x_n$. Every element of $G - \{e\}$, the identity, fixes exactly 2 poles, while the identity fixes them all.

• Here, we use the Counting Theorem:
  • The number of distinct orbits of group $G$ acting on set $X$ is equal to
    \[
    \frac{1}{|G|} \sum_{g \in G} |X^g|
    \]
  • Where $|X^g|$ is the number of elements of $X$ left fixed by group element $G$.
  • For this group action, using the property above, we have:
    \[
    N = \frac{1}{|G|} \{2(|G| - 1) + |X|\} \]
PROOF (BOUNDS ON $N$)

• With some algebraic manipulation of the last expression, we are left with:

$$2 \left(1 - \frac{1}{|G|}\right) = \sum_{i=1}^{N} \left(1 - \frac{1}{|\text{stabilizer}(x_i)|}\right)$$

• Assuming $G$ is not the trivial group $\{e\}$ of size 1, the left size of the equation must be greater than or equal to 1 and less than 2, since $|G| > 1$.

• In addition, each stabilizer has order at least 2 (the poles), so each term of the sum on the right is greater than or equal to $\frac{1}{2}$ and less than 1.

• Thus, $N$ is either 2 or 3.
PROOF (CASES)

• If $N = 2$, then we have $2 = |G(x_1)| + |G(x_2)|$, and there can only be 2 poles. Every element in $G$ must therefore rotate around the axis formed by these two poles, and the plane perpendicular to this axis is mapped onto itself. Therefore, $G$ is isomorphic to a rotation in 2 dimensions and is a cyclic group.

• If $N = 3$, we have through some algebraic manipulation:

$$1 + \frac{2}{|G|} = \frac{1}{|G_x|} + \frac{1}{|G_y|} + \frac{1}{|G_z|}.$$ 

• Note that the terms $|G_x|$, $|G_y|$, and $|G_z|$ must be integers and that the right hand side of the equation must be greater than 1, so we have a set of possible solutions:

(a) $\frac{1}{2}, \frac{1}{2}, \frac{1}{n}$ where $n \geq 2$;
(b) $\frac{1}{2}, \frac{1}{3}, \frac{1}{3}$;
(c) $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$;
(d) $\frac{1}{2}, \frac{1}{3}, \frac{1}{5}$. 
PROOF (MORE CASES)

• By continuing through the casework, we have the following:
  • If we are in situation (a), with $1/2, 1/2, 1/n$, we have a dihedral group.
  • In situation (b), with $1/2, 1/3, 1/3$, we have a regular tetrahedron.
  • In situation (c), with $1/2, 1/3, 1/4$, we have the vertices of a regular octahedron,
    and equivalently the faces of a cube.
  • In situation (d), with $1/2, 1/3, 1/5$, we have the vertices of a regular icosahedron,
    and equivalently the faces of a dodecahedron.

• These are therefore all of the finite subgroups of the group of rotations on 3 dimensions.
• Jacky Chong, mentor
• Naïve Lie Theory, by John Stillwell, reference textbook
• Groups and Symmetry, by M. A. Armstrong, reference textbook, images