An Introduction to Schemes

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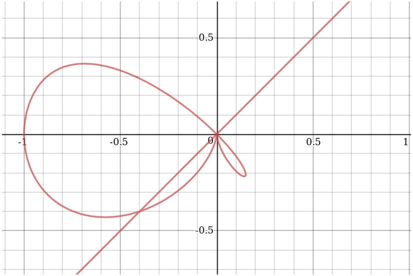
Let k be an algebraically closed field.

Definition

For some set of polynomials $\{f_i\}_{i \in I} \subseteq k[x_1, \ldots, x_n]$, define:
$V(\{f_i\}_{i \in I}) := \{(a_1, \ldots, a_n) \in \mathbb{A}^n \mid f_i(a_1, \ldots, a_n) = 0\}$
A set of this form is called an affine **algebraic set**.



Affine plane curve $V(X^4 - X^2Y^2 + X^5 - Y^5)$ in $\mathbb{A}^2_{\mathbb{R}}$

Ingredients of a Scheme

We want to generalize these varieties to:

1. Handle Non-Algebraically closed fields (And rings in general)
2. Handle multiplicities
3. Connect affine and projective varieties

We need:

1. A set of "points"
2. A topology on these points
3. Functions on the open sets of these points

We'll use the scheme version of the affine line $\mathbb{A}^1_{\mathbb{C}}$ and the integers $\mathbb{Z}$ as examples.

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The "Points"

Take a commutative ring R with unity.

Definition

The **spectrum** of R, $\text{Spec}(R)$, is the set of prime ideals $\mathfrak{p}$ of R.

These serve as our points.

Example: The ring used for $\mathbb{A}^1_{\mathbb{C}}$ is $\mathbb{C}[x]$.
$\mathbb{C}[x]$ has prime ideals of the form $(x - c)$ for $c \in \mathbb{C}$ and the zero ideal (0).

$\mathbb{Z}$ has prime ideals (p) where p is a prime.
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If prime ideals are the points, then what are the elements of R?

They are functions on Spec(R):

Definition

For any $\mathfrak{p} \in \text{Spec}(R)$ and $f \in R$, define the "evaluation of f at $\mathfrak{p}$" to be $f + \mathfrak{p} \in R/\mathfrak{p}$.

$\mathbb{C}[x]$ example: Take $f(x) = x^2 + x + 1 \in \mathbb{C}[x]$ and $(x - 2) \in \mathbb{C}[x]$. Then $x^2 + x + 1 \equiv x - 7$ in $\mathbb{C}[x]/(x - 2)$.

Notice $f(2) = 7$.

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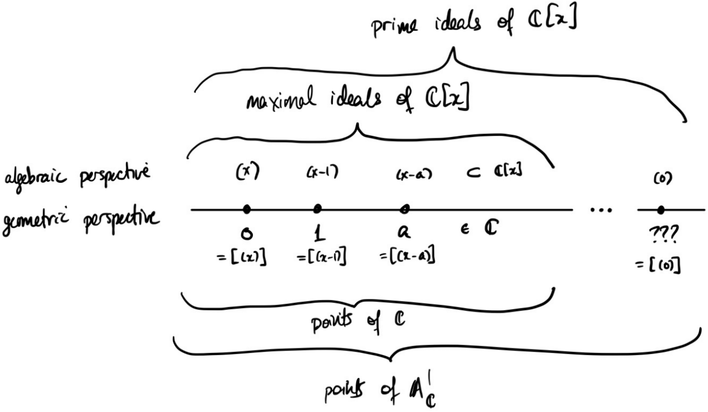
A picture of Spec($\mathbb{C}[x]$)



Visualization of $\mathbb{A}^1_{\mathbb{C}}$ [Vakil]. Note the "generic point" (0) off to the side.

The Zariski Topology

$f \in R$ "evaluating" to 0 at $\mathfrak{p} \in \mathrm{Spec}(R)$ means $f \in \mathfrak{p}$.

Definition

Given any subset $S \subseteq R$, define $V(S) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} \supseteq S\}$

$\mathbb{C}[x]$ example: $V(x^2 + x - 6) = \{[(x-2)], [(x+3)]\}$.
Notice this is just finding the roots of $x^2 + x - 6$.

Definition

The sets $V(S)$ for all $S \subseteq R$ satisfy the axioms for being the closed sets of a topology. We define this as the **Zariski topology** on $\mathrm{Spec}(R)$.

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Distinguished open sets

We have a nice basis for the Zariski topology:

Definition

For any $f \in R$, define the **distinguished open set** $D(f) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid f \notin \mathfrak{p}\} = \mathrm{Spec}(R) \setminus V(f)$

$\mathbb{Z}$ example: $D(6)$ is set of all "primes" p such that $6 \not\equiv 0 \mod p$.
This means $D(6) = \{(p) \in \mathrm{Spec}(\mathbb{Z}) \mid p \nmid 6\}$.

Theorem

The distinguished open sets form a basis for the Zariski topology on $\mathrm{Spec}(R)$.

We use these to define the structure sheaf on $\mathrm{Spec}(R)$.

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Presheaves

The last piece of a scheme is its structure sheaf.
We want something like functions on open sets of a manifold.

Definition

Given a topological space X a sheaf $\mathscr{F}$ assigns for each open set U of X a set (group, ring, etc) $\mathscr{F}(U)$. This can be seen as the "set of functions on U". We then want:

1. If $V \subseteq U$ are open sets in X, we can "restrict" a function $f \in \mathscr{F}(U)$ uniquely to some $f' \in \mathscr{F}(V)$.
2. If $\{f_i\}_{i \in I}$ is a set of functions each defined on U_i that agree on interlaps, we want to be able to "glue together" the f_i's to some $f \in \mathscr{F}(\bigcap_{i \in I} U_i)$.
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The Structure Sheaf

Going back now to $\text{Spec}(R)$ with its Zariski topology, the structure sheaf is a sheaf of rings on $\text{Spec}(R)$.

Definition

For each distinguished open set $D(f) \subseteq \text{Spec}(R)$, Define: $\mathscr{O}_{\text{Spec}(R)}(D(f))$ to be the localization of R at the set $S = \{g \in R \mid V(g) \subseteq V(f)\}$, which is isomorphic to R_S.

We can think of this as "rational functions" with the denominator not vanishing where f vanishes.

$\mathbb{C}[x]$ example: If we take $x \in \mathbb{C}[x]$ then $\mathscr{O}_{\text{Spec}(\mathbb{C}[x])}(D(x)) = \mathbb{C}[x]_x$, which is rational functions f/g where $f, g \in \mathbb{C}[x]$ $x \nmid g$.

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General Schemes

The spectrum of R, the Zariski topology on Spec(R) and the structure sheaf on the topological space give an **Affine Scheme**.

More general schemes are constructed by gluing affine schemes together:

Definition

A **Scheme** is a topological space X with a sheaf of rings where for every point $p \in X$ there is a neighborhood U of p such that $U \cong \text{Spec}(R)$ for some ring R. *

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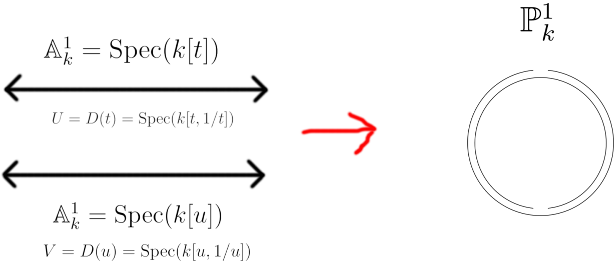
Example: Projective Line

We can construct the projective line by gluing together two affine lines.



$$U \to V \quad t \mapsto 1/u$$

The gluing of the Affine lines [Vakil].

The Projective Line Continued

Theorem

$\mathbb{P}^1_k$ is not isomorphic to the spectrum of any ring, that is $\mathbb{P}^1_k$ is not an affine scheme.

This is because if $\mathbb{P}^1_k$ was affine then $\mathbb{P}^1_k$ would be isomorphic to the spectrum of the ring of "global sections" over $\mathbb{P}^1_k$.
But the only polynomials defined over all of $\mathbb{P}^1_k$ are constant, thus $\mathrm{Spec}(\Gamma(\mathbb{P}^1_k, \mathscr{O}_{\mathbb{P}^1_k})) \cong \mathrm{Spec}(k)$, which is only one point: [(0)].