An Introduction to Schemes

Nicholas Hiebert-White
Advisor: Patrick Daniels

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Let $k$ be an algebraically closed field.

**Definition**

For some set of polynomials $\{f_i\}_{i \in I} \subseteq k[x_1, \ldots, x_n]$, define:

$$V(\{f_i\}_{i \in I}) := \{(a_1, \ldots, a_n) \in \mathbb{A}^n \mid f_i(a_1, \ldots, a_n) = 0\}$$

A set of this form is called an affine **algebraic set**.

Affine plane curve $V(X^4 - X^2 Y^2 + X^5 - Y^5)$ in $\mathbb{A}^2_{\mathbb{R}}$
We want to generalize these varieties to:

1. Handle Non-Algebraically closed fields (And rings in general)
2. Handle multiplicities
3. Connect affine and projective varieties

We need:

1. A set of “points”
2. A topology on these points
3. Functions on the open sets of these points

We’ll use the scheme version of the affine line $\mathbb{A}^1_{\mathbb{C}}$ and the integers $\mathbb{Z}$ as examples.
Ingredients of a Scheme

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Take a commutative ring $R$ with unity.

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The **spectrum** of $R$, $\text{Spec}(R)$, is the set of prime ideals $p$ of $R$.

These serve as our points.

Example: The ring used for $\mathbb{A}^1_{\mathbb{C}}$ is $\mathbb{C}[x]$. $\mathbb{C}[x]$ has prime ideals of the form $(x - c)$ for $c \in \mathbb{C}$ and the zero ideal $(0)$.

$\mathbb{Z}$ has prime ideals $(p)$ where $p$ is a prime.

$(p) = \{ ap \mid a \in \mathbb{Z}\}$
The “Points”

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\( \mathbb{C}[x] \) has prime ideals of the form \((x - c)\) for \( c \in \mathbb{C} \) and the zero ideal \((0)\).

\( \mathbb{Z} \) has prime ideals \((p)\) where \( p \) is a prime.
\((p) = \{ap \mid a \in \mathbb{Z}\}\)
If prime ideals are the points, then what are the elements of \( R \)?

They are functions on \( \text{Spec}(R) \):

**Definition**
For any \( p \in \text{Spec}(R) \) and \( f \in R \), define the “evaluation of \( f \) at \( p \)” to be \( f + p \in R/p \).

\( \mathbb{C}[x] \) example: Take \( f(x) = x^2 + x + 1 \in \mathbb{C}[x] \) and \( (x - 2) \in \mathbb{C}[x] \). Then \( x^2 + x + 1 \equiv x - 7 \) in \( \mathbb{C}[x]/(x - 2) \).

Notice \( f(2) = 7 \).

\( \mathbb{Z} \) example: Take \( (5) \) in \( \text{Spec}(\mathbb{Z}) \). Then 11 “evaluated at” \( (5) \) is \( 11 \equiv 1 \mod 5 \).
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A picture of Spec(\(\mathbb{C}[x]\))

Visualization of \(A^{1}_{\mathbb{C}}\) [Vakil]. Note the “generic point” (0) off to the side.
$f \in R$ “evaluating” to 0 at $p \in \text{Spec}(R)$ means $f \in p$.

**Definition**

Given any subset $S \subseteq R$, define $V(S) = \{p \in \text{Spec}(R) \mid p \supseteq S\}$

$\mathbb{C}[x]$ example: $V(x^2 + x - 6) = \{[(x - 2)], [(x + 3)]\}$. Notice this is just finding the roots of $x^2 + x - 6$.

**Definition**

The sets $V(S)$ for all $S \subseteq R$ satisfy the axioms for being the closed sets of a topology. We define this as the **Zariski topology** on $\text{Spec}(R)$.
The Zariski Topology

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We have a nice basis for the Zariski topology:

**Definition**
For any \( f \in R \), define the **distinguished open set**
\[
D(f) = \{ p \in \text{Spec}(R) \mid f \not\in p \} = \text{Spec}(R) \setminus V(f)
\]

\( \mathbb{Z} \) example: \( D(6) \) is set of all “primes” \( p \) such that \( 6 \not\equiv 0 \pmod{p} \).
This means \( D(6) = \{ (p) \in \text{Spec}(\mathbb{Z}) \mid p \nmid 6 \} \).

**Theorem**
The distinguished open sets form a basis for the Zariski topology on \( \text{Spec}(R) \).

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The last piece of a scheme is its structure sheaf. We want something like functions on open sets of a manifold.

**Definition**

Given a topological space $X$ a sheaf $\mathcal{F}$ assigns for each open set $U$ of $X$ a set (group, ring, etc) $\mathcal{F}(U)$. This can be seen as the "set of functions on $U". We then want:

1. If $V \subseteq U$ are open sets in $X$, we can "restrict" a function $f \in \mathcal{F}(U)$ uniquely to some $f' \in \mathcal{F}(V)$.
2. If $\{f_i\}_{i \in I}$ is a set of functions each defined on $U_i$ that agree on interlaps, we want to be able to "glue together" the $f_i'$s to some $f \in \mathcal{F}(\bigcap_{i \in I} U_i)$.
3. We want the above gluing to be unique.
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Going back now to $\text{Spec}(R)$ with its Zariski topology, the structure sheaf is a sheaf of rings on $\text{Spec}(R)$.

**Definition**

For each distinguished open set $D(f) \subseteq \text{Spec}(R)$, Define:

$\mathcal{O}_{\text{Spec}(R)}(D(f))$ to be the localization of $R$ at the set $S = \{ g \in R \mid V(g) \subseteq V(f) \}$, which is isomorphic to $R_S$.

We can think of this as “rational functions” with the denominator not vanishing where $f$ vanishes.

$\mathbb{C}[x]$ example: If we take $x \in \mathbb{C}[x]$ then $\mathcal{O}_{\text{Spec}(\mathbb{C}[x])}(D(x)) = \mathbb{C}[x]_x$, which is rational functions $f/g$ where $f, g \in \mathbb{C}[x]$ $x \nmid g$.

We can then extend this definition to get a sheaf on all open sets of $\text{Spec}(R)$. 
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The spectrum of $R$, the Zariski topology on $\text{Spec}(R)$ and the structure sheaf on the topological space give an **Affine Scheme**.

More general schemes are constructed by gluing affine schemes together:

**Definition**

A **Scheme** is a topological space $X$ with a sheaf of rings where for every point $p \in X$ there is a neighborhood $U$ of $p$ such that $U \cong \text{Spec}(R)$ for some ring $R$. *

* This isomorphism is as *ringed spaces*, which roughly means the sheaves are isomorphic also.
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Example: Projective Line

We can construct the projective line by gluing together two affine lines.

\[ \mathbb{A}^1_k = \text{Spec}(k[t]) \]

\[ U = D(t) = \text{Spec}(k[t, 1/t]) \]

\[ \mathbb{A}^1_k = \text{Spec}(k[u]) \]

\[ V = D(u) = \text{Spec}(k[u, 1/u]) \]

\[ U \rightarrow V \quad t \mapsto 1/u \]

The gluing of the Affine lines [Vakil].
The Projective Line Continued

Theorem

$\mathbb{P}^1_k$ is not isomorphic to the spectrum of any ring, that is $\mathbb{P}^1_k$ is not an affine scheme.

This is because if $\mathbb{P}^1_k$ was affine then $\mathbb{P}^1_k$ would be isomorphic to the spectrum of the ring of “global sections” over $\mathbb{P}^1_k$. But the only polynomials defined over all of $\mathbb{P}^1_k$ are constant, thus $\text{Spec}(\Gamma(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k})) \cong \text{Spec}(k)$, which is only one point: $[(0)]$. 