Riemann’s Inequality for Algebraic Curves and its Consequences

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Affine Plane Curves

Definition

Let $k$ be any field. The \textbf{affine $n$-space} over $k$ is defined as

$$A^n(k) := \{(x_1, x_2, \ldots, x_n) \mid x_i \in k\}$$
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The **affine plane** over a field $k$ is defined as

$$\mathbb{A}^2(k) := \{(x_1, x_2) | x_1, x_2 \in k\}$$
Affine Plane Curves

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**Definition**

An **affine plane curve** $C$ is a set of form

$$C := \{(x, y) \in \mathbb{A}^2(k) \mid F(x, y) = 0\}$$

where $F(x, y) \in k[x, y]$
We want a meaningful way to talk about the "intersection" of any two curves.
Shortcomings of $\mathbb{A}^n$

- We want a meaningful way to talk about the "intersection" of any two curves.
- We want to "enlarge" the plane such that any two curves will "intersect" at some "point."
The **projective n-space** $\mathbb{P}^n$ over $k$ is the set of all equivalence classes of points in $\mathbb{A}^{n+1} \setminus \{(0,0,\ldots,0)\}$ such that

$$(a_1, a_2, \ldots a_{n+1}) \equiv (\lambda a_1, \lambda a_2, \ldots \lambda a_{n+1})$$

for all $\lambda \in k$, $\lambda \neq 0$. 

---

**Definition**

A **projective plane curve** $C$ is a set

$$C := \{[x:y:z] \in \mathbb{P}^2 | F(x,y,z) = 0\}$$

where $F(x,y,z)$ is a form in $k[x,y,z]$. 

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Riemann's Inequality
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$$C := \{[x : y : z] \in \mathbb{P}^2 \mid F(x, y, z) = 0\}$$

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A projective plane curve $C$ is **irreducible** if it is the zero set of an irreducible form. The curve is **reducible** otherwise.
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Suppose $C$ is an affine plane curve determined by the polynomial $F$. A point $P$ on $C$ is a **simple** point if either $F_x(P) \neq 0$ or $F_y(P) \neq 0$. Otherwise we say $P$ is a **singular** point.
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Definition

Suppose $C$ is a projective plane curve determined by a form polynomial $F$. A point $P$ on $C$ is **simple** if the affine plane curve determined by dehomogenized polynomial $F_*$ is simple at the analogous point. Otherwise we say that $P$ is **singular**. We say $C$ is **nonsingular** if all points are simple.
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- Our field $k$ will be algebraically closed field of characteristic 0.
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- The field of rational functions on $C$ will be notated $K$. 

Proposition 1: At a point $P$ on $C$, every nonzero $z \in K$ can be expressed uniquely as $z = ut^n$, where $u$ is a unit in the local ring of $C$ at $P$ and $t$ is a fixed irreducible element in the local ring, called the uniformizing parameter, with $n \in \mathbb{Z}$. We say that $n$ is the order of $z$ at $P$ on $C$. 

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A **divisor** $D$ on $C$ is a formal sum

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Definition

The **degree** of a divisor $D$ is the sum of its coefficients, i.e.

$$\text{deg}(D) := \sum_{P \in C} n_P$$

A divisor $D$ is **effective** if each $n_P \geq 0$, and we write $\sum n_P P \geq \sum m_P P$ if each $n_P \geq m_P$. 
Definition

For any nonzero \( z \in K \), define the divisor of \( z \) as

\[
div(z) = \sum_{P \in C} \text{ord}_P(z)P
\]
Divisors (cont.)

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\]

**Definition**

We define the **divisor of zeros of** \( z \) as

\[
(z)_0 = \sum_{\ord_P(z) > 0} \ord_P(z)P
\]

and we define the **divisor of poles of** \( z \) as

\[
(z)_\infty = \sum_{\ord_P(z) < 0} \ord_P(z)P
\]
Remark

The set of divisors on $C$ form the free abelian group on the set of points of $C$ under formal addition.
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**Definition**

Two divisors $D$ and $D'$ are **linearly equivalent** if $D' = D + \text{div}(z)$ for some $z \in K$, in which case we write $D' \equiv D$. 

**Proposition 2**

(i) The relation $\equiv$ is an equivalence relation.

(ii) $D \equiv 0$ if and only if $D = \text{div}(z)$ for some $z \in K$.

(iii) If $D \equiv D'$, then $\deg(D) = \deg(D')$.

(iv) If $D \equiv D'$ and $D_1 \equiv D_1'$, then $D + D_1 \equiv D' + D_1'$. 


Properties of Divisors

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The Vector Spaces $L(D)$

**Definition**

Let $D = \sum n_P P$ be a divisor on $C$. We define

$L(D) := \{ f \in K \mid \text{ord}_P(f) \geq -n_P \text{ for all } P \in C \}$. 

**Remark**

$L(D)$ forms a vector space over $k$.

**Definition**

The dimension of $L(D)$ over $k$ is denoted $l(D)$. 

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Proposition 3

Let $D$ and $D'$ be divisors on $C$.

(i) If $D \leq D'$, then $L(D) \subset L(D')$ and

$\dim_k(L(D')/L(D)) \leq \deg(D' - D)$

(ii) $L(0) = k$; $L(D) = 0$ if $\deg(D) < 0$

(iii) $L(D)$ is finite dimensional for all $D$. If $\deg(D) \geq 0$, then

$l(D) \leq \deg(D) + 1$

(iv) If $D \equiv D'$, then $l(D) = l(D')$
How ”big” is $L(D)$? Can we determine $I(D)$ exactly only using properties of $D$ and $C$?
In fact, we can! The following Lemma answers part of the question for divisors of a special form.

Lemma

Let \( x \in K, x \notin k \). Let \( Z = (x)_0 \) be the divisor of zeros of \( x \) and let \( n = [K : k(x)] \). Then:

(i) \( Z \) is an effective divisor of degree \( n \)
(ii) There is a constant \( \tau \) such that \( l(rZ) \geq rn - \tau \) for all \( r \)
More generally, we observe that $l(D)$ is "bounded" below, specifically determined by properties of $D$ and $C$. 
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**Theorem**

There is an integer $g$ such that

$$l(D) \geq \deg(D) + 1 - g$$

for all divisors $D$ on $C$. The smallest such $g$ is called the **genus** of $C$. The genus must be a nonnegative integer.
Proof Sketch

For each $D = \sum m_P P$, let $s(D) = \text{deg}(D) + 1 - l(D)$. We want $g$ such that $s(D) \leq g$ for all $D$. 
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• Observe $s(0) = 0$, so $g \geq 0$ if it exists, by well-ordering principle.
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If $D \equiv D'$, then $s(D) = s(D')$. 
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- If $D \equiv D'$, then $s(D) = s(D')$.
- If $D \leq D'$, then $s(D) \leq s(D')$.
- Let $\alpha \in K, \alpha \notin k$. Let $Z = (\alpha)_0$. By Lemma, there exists smallest $\tau$ such that $l(rZ) \geq rn - \tau$ for all $r$. 

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- If $D \leq D'$, then $s(D) \leq s(D')$.
- Let $x \in K, x \notin k$. Let $Z = (x)_0$. By Lemma, there exists smallest $\tau$ such that $l(rZ) \geq rn - \tau$ for all $r$.
- After some algebra and using properties of $l(D)$ and $\text{deg}(D)$, we see that $s(rZ) = \tau + 1$ for all large $r > 0$. Let $g = \tau + 1$. 

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• Let $x \in K$, $x \notin k$. Let $Z = (x)_0$. By Lemma, there exists smallest $\tau$ such that $l(rZ) \geq rn - \tau$ for all $r$.

• After some algebra and using properties of $l(D)$ and $\deg(D)$, we see that $s(rZ) = \tau + 1$ for all large $r > 0$. Let $g = \tau + 1$.

• Then it suffices to find a divisor $D'$ such that $D \equiv D'$ and an integer $r \geq 0$ such that $D' \leq rZ$. 

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Riemann's Inequality
Recall $D = \sum m_P P$ and let $Z = \sum n_P P$. We want $m_P - \text{ord}(f) \leq r_P$ for all $P$, because this gives $D' = D - \text{div}(f)$ with the desired properties.

Let $y = x - 1$. Let $T = \{P \in C | m_P > 0 \text{ and } \text{ord}_P(y) \geq 0\}$.

Let $f = \prod_{P \in T} (y - y(P))^{m_P}$.

Observe that $m_P - \text{ord}_P(f) \leq 0$ when $\text{ord}_P(y) \geq 0$, so this satisfies what we want. If $\text{ord}_P(y) < 0$, then $n_P > 0$, so we can just choose a large $r$ to satisfy the inequalities we want. This proves the theorem.
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This proves the theorem.
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If \( l(D_0) = \deg(D_0) + 1 - g \) and \( D \equiv D' \geq D_0 \), then \( l(D) = \deg(D) + 1 - g \).
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If \( x \in K, x \notin k \), then \( g = \deg(r(x)_0) - l(r(x)_0) + 1 \) for all sufficiently large \( r \).
Corollaries

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Corollary 2
If $x \in K$, $x \notin k$, then $g = \deg(r(x)_0) - l(r(x)_0) + 1$ for all sufficiently large $r$.

Corollary 3
There is an integer $N$ such that for all divisors $D$ of degree greater than $N$, we have $l(D) = \deg(D) + 1 - g$. 
Now that we have "bounded" $l(D)$ from below, can we do the same from above? In other words, is there a way to determine $l(D)$ exactly, not just in terms of inequality?
Yes we can!

The "other side" of the inequality was resolved by Riemann's student Gustav Roch in 1865. The final result is the famous Riemann-Roch Theorem.

Remark
There is a special type of divisor $W$ on $\mathcal{C}$ of degree $2g - 2$ called a Canonical Divisor.

Theorem
Let $W$ be a canonical divisor on $\mathcal{C}$. Let the genus of $\mathcal{C}$ be $g$. Then for any divisor $D$,

$$l(D) = \deg(D) + 1 - g + l(W - D).$$
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Fulton, William (2008)

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