Riemann's Inequality for Algebraic Curves and its Consequences

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May 8, 2019

Affine Plane Curves

Definition

Let k be any field. The **affine n-space** over k is defined as

$$\mathbb{A}^n(k) := \{(x_1, x_2, \ldots, x_n) \mid x_i \in k\}$$

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Definition

An **affine plane curve** C is a set of form

$$C := \{(x, y) \in \mathbb{A}^2(k) \mid F(x, y) = 0\}$$

where $F(x, y) \in k[x, y]$

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- We want a meaningful way to talk about the "intersection" of any two curves
- We want to "enlarge" the plane such that any two curves will "intersect" at some "point."

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The **projective n-space** $\mathbb{P}^n$ over k is the set of all equivalence classes of points in $\mathbb{A}^{n+1} \setminus \{(0, 0, \dots, 0)\}$ such that $(a_1, a_2, \dots a_{n+1}) \equiv (\lambda a_1, \lambda a_2, \dots \lambda a_{n+1})$ for all $\lambda \in k$, $\lambda \neq 0$

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A **projective plane curve** C is a set

$$C := \{[x : y : z] \in \mathbb{P}^2 \mid F(x, y, z) = 0\}$$

where $F(x, y, z)$ is a form in $k[x, y, z]$

Projective Plane Curves (cont.)

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A projective plane curve C is **irreducible** if it is the zero set of an irreducible form. The curve is **reducible** otherwise.

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Definition

Suppose C is an affine plane curve determined by the polynomial F. A point P on C is a **simple** point if either $F_x(P) \neq 0$ or $F_y(P) \neq 0$. Otherwise we say P is a **singular** point.

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Suppose C is a projective plane curve determined by a form polynomial F. A point P on C is **simple** if the affine plane curve determined by dehomogenized polynomial F_* is simple at the analogous point. Otherwise we say that P is **singular**. We say C is **nonsingular** if all points are simple.

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- Our field k will be algebraically closed field of characteristic 0.

Algebraic Preliminaries and Review

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Proposition 1

At a point P on C, every nonzero $z \in K$ can be expressed uniquely as $z = ut^n$, where u is a unit in the local ring of C at P and t is a fixed irreducible element in the local ring, called the **uniformizing parameter**, with $n \in \mathbb{Z}$. We say that n is the **order** of z at P on C.

Divisors

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$$D := \sum_{P \in C} n_P P$$

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The **degree** of a divisor D is the sum of its coefficients, i.e.

$$deg(D) := \sum_{P \in C} n_P$$

A divisor D is **effective** if each $n_P \geq 0$, and we write $\sum n_P P \geq \sum m_P P$ if each $n_P \geq m_P$.

Definition

For any nonzero $z \in K$, define the **divisor of** z as

$$div(z) = \sum_{P \in C} ord_P(z)P$$

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We define the **divisor of zeros of** z as

$$(z)_0 = \sum_{ord_P(z)>0} ord_P(z)P$$

and we define the **divisor of poles of** z as

$$(z)_\infty = \sum_{ord_P(z)<0} ord_P(z)P$$

Properties of Divisors

Remark

The set of divisors on C form the free abelian group on the set of points of C under formal addition.

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Proposition 2

(i) The relation $\equiv$ is an equivalence relation
(ii) $D \equiv 0$ if and only if $D = div(z)$ for some $z \in K$
(iii) If $D \equiv D'$, then $deg(D) = deg(D')$
(iv) If $D \equiv D'$ and $D_1 \equiv D_1'$, then $D + D_1 \equiv D' + D_1'$

The Vector Spaces $L(D)$

Definition

Let $D = \sum n_P P$ be a divisor on C. We define $L(D) := \{f \in K \mid ord_P(f) \geq -n_P \text{ for all } P \in C\}$.

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Definition

The dimension of $L(D)$ over k is denoted $l(D)$.

Properties of $L(D)$

Proposition 3

Let D and D' be divisors on C.
(i) If $D \leq D'$, then $L(D) \subset L(D')$ and $dim_k(L(D')/L(D)) \leq deg(D' - D)$
(ii) $L(0) = k$; $L(D) = 0$ if $deg(D) < 0$
(iii) $L(D)$ is finite dimensional for all D. If $deg(D) \geq 0$, then $l(D) \leq deg(D) + 1$
(iv) If $D \equiv D'$, then $l(D) = l(D')$

Motivating Question

How "big" is $L(D)$? Can we determine $l(D)$ exactly only using properties of D and C?

Lemma

In fact, we can! The following Lemma answers part of the question for divisors of a special form.

Lemma

Let $x \in K$, $x \notin k$. Let $Z = (x)_0$ be the divisor of zeros of x and let $n = [K : k(x)]$. Then:
(i)Z is an effective divisor of degree n
(ii) There is a constant τ such that $l(rZ) \geq rn - \tau$ for all r

Riemann's Inequality

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Theorem

There is an integer g such that

$$l(D) \geq deg(D) + 1 - g$$

for all divisors D on C. The smallest such g is called the **genus** of C. The genus must be a nonnegative integer.

Proof Sketch

- For each $D = \sum m_P P$, let $s(D) = deg(D) + 1 - l(D)$. We want g such that $s(D) \leq g$ for all D.

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- After some algebra and using properties of $l(D)$ and $deg(D)$, we see that $s(rZ) = \tau + 1$ for all large $r > 0$. Let $g = \tau + 1$.

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- After some algebra and using properties of $l(D)$ and $deg(D)$, we see that $s(rZ) = \tau + 1$ for all large $r > 0$. Let $g = \tau + 1$.
- Then it suffices to find a divisor D' such that $D \equiv D'$ and an integer $r \geq 0$ such that $D' \leq rZ$.

Proof Sketch (cont.)

- Recall $D = \sum m_P P$ and let $Z = \sum n_P P$.

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- If $ord_P(y) < 0$, then $n_P > 0$, so we can just choose a large r to satisfy the inequalities we want.
- This proves the theorem.

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If $l(D_0) = deg(D_0) + 1 - g$ and $D \equiv D' \geq D_0$, then $l(D) = deg(D) + 1 - g$.

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Corollary 2

If $x \in K$, $x \notin k$, then $g = deg(r(x)_0) - l(r(x)_0) + 1$ for all sufficiently large r.

Corollary 3

There is an integer N such that for all divisors D of degree greater than N, we have $l(D) = deg(D) + 1 - g$.

Now that we have "bounded" $I(D)$ from below, can we do the same from above? In other words, is there a way to determine $I(D)$ exactly, not just in terms of inequality?

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There is a special type of divisor W on C of degree $2g - 2$ called a **Canonical Divisor**.

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There is a special type of divisor W on C of degree $2g-2$ called a **Canonical Divisor**.

Theorem

Let W be a canonical divisor on C. Let the genus of C be g. Then for any divisor D,

$$l(D) = deg(D) + 1 - g + l(W - D)$$

References

Fulton, William (2008)
Algebraic Curves: An Introduction to Algebraic Geometry