EXPLORING TRANSCENDENTAL EXTENSIONS

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FIELD

Definition: A *field* is a non-empty set F with two binary operations on F, namely, "+" (addition) and "·" (multiplication), satisfying the following *field axioms*:

Property	Addition	Multiplication
Closure	$x + y \in F$, for all $x, y \in F$	$x \cdot y \in F$, for all $x, y \in F$
Commutativity	$x + y = y + x$, for all $x, y \in F$	$x \cdot y = y \cdot x$, for all $x, y \in F$
Associativity	$(x + y) + z = x + (y + z)$, for all $x, y, z \in F$	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$, for all $x, y, z \in F$
Identity	There exists an element $0 \in F$ such that $0 + x = x + 0 = x$, for all $x \in F$ (Additive Identity)	There exists an element $1 \in F$ such that $1 \cdot x = x \cdot 1 = x$, for all $x \in F$ (Multiplicative Identity)
Inverse	For all $x \in F$, there exists $y \in F$ such that $x + y = 0$ (Additive Inverse)	For all $x \in F^{\times}$, there exists $y \in F$ such that $x \cdot y = 1$ (Multiplicative Inverse)
Distributivity (Multiplication is distributive over addition)	For all $x, y, z \in F$, $x \cdot (y + z) = x \cdot y + x \cdot z$	

EXAMPLES

- Set of Real Numbers, $\mathbb{R}$
- Set of Complex Numbers, $\mathbb{C}$
- Set of Rational Numbers, $\mathbb{Q}$
- $\mathbb{F}_2 = \{0,1\} = \mathbb{Z}/2\mathbb{Z}$
- In general, $\mathbb{F}_p = \{0,1,\ldots,p-1\} = \frac{\mathbb{Z}}{p\mathbb{Z}}$, where p is prime
- $\mathbb{C}(X)$, the field of rational functions with complex coefficients
- $\mathbb{R}(X)$, the field of rational functions with real coefficients
- $\mathbb{Q}(X)$, the field of rational functions with rational coefficients
- In general, $K(X)$, where K is a field

EXTENSION FIELDS

Definition: A field E containing a field F is called an *extension field* of F (or simply an extension of F, denoted by E/F). Such an E is regarded as an F–vector space. The dimension of as an F–vector space is called the degree of E over F and is denoted by $[E: F]$. We say E is *finite* over F (or a finite extension of F) if it has a finite degree over F and *infinite* otherwise.

Examples: (a) The field of complex numbers, $\mathbb{C}$, is a finite extension of $\mathbb{R}$ and has degree 2 over $\mathbb{R}$ (basis $\{1, i\}$)
(b) The field of real numbers, $\mathbb{R}$, has an infinite degree over the field of rationals, $\mathbb{Q}$: the field $\mathbb{Q}$ is countable, and so every finite-dimensional $\mathbb{Q}$–vector space is also countable, but a famous argument of Cantor shows that $\mathbb{R}$ is not countable.
(c) The field of Gaussian rationals, $\mathbb{Q}(i) = \{a + bi: a, b \in \mathbb{Q}\}$, has degree 2 over $\mathbb{Q}$ (basis $\{1, i\}$)
(d) The field $F(X)$ has infinite degree over F; in fact, even its subspace $F[X]$ has infinite dimension over F

ALGEBRAIC AND TRANSCENDENTAL ELEMENTS

Definition: An element α in E is *algebraic* over F, if $\mathrm{f}(\alpha)=0$, for some non-zero polynomial $\mathrm{f}\in\mathrm{F}[\mathrm{X}]$. An element that is not algebraic over F is *transcendental* over F.

Examples: (a) The number $\alpha=\sqrt{2}$ is algebraic over $\mathbb{R}$ since $\mathrm{p}(\sqrt{2})=0$, for $\mathrm{p}(\mathrm{X})=\mathrm{X}^2-2\in\mathbb{R}[\mathrm{X}]$

(b) The number $\alpha=\sqrt[3]{3}$ is algebraic over $\mathbb{Q}$ since $\mathrm{h}(\sqrt[3]{3})=0$, for $\mathrm{h}(\mathrm{X})=\mathrm{X}^3-3\in\mathbb{Q}[\mathrm{X}]$

(c) The number $\pi=3.141\ldots$ is transcendental over $\mathbb{Q}$

(d) The number $\alpha=\pi$ is algebraic over $\mathbb{Q}(\pi)$ since $\mathrm{q}(\pi)=0$ for $\mathrm{q}(\mathrm{X})=\mathrm{X}-\pi\in\mathbb{Q}(\pi)[\mathrm{X}]$

ALGEBRAIC AND TRANSCENDENTAL EXTENSIONS

Definition: A field extension E/F is said to be an *algebraic extension*, and E is said to be algebraic over F, if all elements of E are algebraic over F. Otherwise, E is transcendental over F. Thus, E/F is *transcendental* if at least one element of E is transcendental over F.

Remark: A field extension E/F is finite if and only if E is algebraic and finitely generated (as a field) over F.

Examples: (a) The field of real numbers is a transcendental extension of the field $\mathbb{Q}$ since π is transcendental over $\mathbb{Q}$

(b) The field $\mathbb{Q}(e)$ is a transcendental extension of $\mathbb{Q}$ since e is transcendental over $\mathbb{Q}$

(c) The field of rational functions $F(X)$ in the variable X is a transcendental extension of the field F since X is transcendental over F.

(d) The field $\mathbb{Q}(\sqrt{2})$ is an algebraic extension of $\mathbb{Q}$ since it has degree 2 (finite) over $\mathbb{Q}$

(e) The field $\mathbb{Q}(\sqrt[3]{3})$ is an algebraic extension of $\mathbb{Q}$ since it has degree 3 (finite) over $\mathbb{Q}$

TRANSCENDENCE BASE

Definition: A subset $S = \{a_1, \dots, a_n\}$ of E is called *algebraically independent* over F if there is no non-zero polynomial $f(x_1, \dots, x_n) \in F[X_1, \dots, X_n]$ such that $f(a_1, \dots, a_n) = 0$. A *transcendence base* for E/F is a maximal subset (with respect to inclusion) of E which is algebraically independent over F.

Note that if E/F is an algebraic extension, the empty set is the only algebraically independent subset of E. In particular, elements of an algebraically independent set are necessarily transcendental.

THEOREM

Theorem: The extension E/F has a transcendence base and any two transcendence bases of E/F have the same cardinality

Remark: The *cardinality of a transcendence base* for E/F is called the *transcendence degree* of E/F. Algebraic extensions are precisely the extensions of transcendence degree 0. Note that if S_1 and S_2 are transcendence bases for E/F, it is not necessarily the case that $F(S_1) = F(S_2)$.

NOETHER'S NORMALIZATION THEOREM

Theorem: Suppose that R is a finitely generated domain over a field K. Then, there exists an algebraically independent subset $\mathcal{L}= \{y_1, \dots, y_r\}$ of R so that R is integral over $R[\mathcal{L}]$

Sketch Of The Proof:

Definition: In commutative algebra, an element b of a commutative ring B *is said to be integral over* A, a subring of B, if b is a root of a monic polynomial over A. If every element of B is integral over A, then B *is said to be integral over* A.

(i) The proof is done by induction on n, the number of generators of R over K. Thus, $R = K[x_1, \dots, x_n]$

(ii) If $n = 0$, then $R = K$ (Nothing to Prove). If $n = 1$, then $R = K[x_1]$. Then, there are two cases:

(a) If x_1 is algebraic, then $r = 0$ and x_1 is integral over K. So, the theorem holds.

(b) If x_1 is transcendental, then set $x_1 = y_1$. Then, we get $R = K[x_1]$, which is integral over $K[x_1]$.

(iii) Now, let $n \geq 2$. If $x_1, \dots, x_n$ are algebraically independent, then set $x_i = y_i, \forall i$ and we're done. If not, then there exists a non-zero polynomial $f(X) \in K[X_1, \dots, X_n]$ such that $f(x_1, \dots, x_n) = 0$.

NOETHER'S NORMALIZATION THEOREM (CONTN.)

The polynomial can be written as $f(X) = \sum_\alpha c_\alpha X^\alpha$, where we use the notation $X^\alpha = X_1^{a_1} \dots X_n^{a_n}$ for $\alpha = (a_1, \dots, a_n)$.

(iv) Rewriting the above polynomial as a polynomial in X_1 with coefficients in $K[X_2, \dots, X_n]$, we have:

$$f(X) = \sum_{j=0}^{N} f_j(X_1, \dots, X_n) X_1^j$$

Since f is non-zero, it involves at least one of the X_i and we can assume it is X_1. Now, we want to somehow arrange to have $f_N = 1$. Then, x_1 would be integral over $K[x_2, \dots, x_n]$, which by induction on n would be integral over $K[\mathcal{L}]$, for some algebraically independent subset $\mathcal{L}$. Since the integral extensions of integral extensions are integral, the theorem follows.

(v) To make f monic, we perform a change of variables that transforms or "normalizes" f into a monic polynomial in X_1. Let $Y_2, \dots Y_n$ and $y_2, \dots, y_n \in R$ be given by $Y_i = X_i - X_1^{m_i}$, where the positive integers $m_i = d^{i-1}$, where d is an integer greater than any of the exponents which occur in the polynomial $f(X)$. This gives us a new polynomial $g(X_1, Y_2, \dots Y_n) = f(X_1, \dots, X_n) \in K[X_1, Y_2, \dots, Y_n]$ such that $g(x_1, y_2, \dots, y_n) = 0$. Then,

$$g(X_1, Y_2, \dots, Y_n) = \sum_\alpha c_\alpha X_1^{a_1} (X_1^d + X_2)^{a_2} (X_1^{d^2} + X_3)^{a_3} \dots (X_1^{d^{n-1}} + X_n)^{a_n}$$

NOETHER'S NORMALIZATION THEOREM (CONTN.)

It is easy to see that $K[X_1, Y_2, \ldots, Y_n] = K[X_1, X_2, \ldots, X_n]$. Now, the highest power of X_1 which occurs is $N = \sum a_i d^{i-1}$. The coefficient of X_1^N is c_α. We can divide g by this non-zero constant and make g monic in X_1 and we're done by induction.

Relevance: Noether's Normalization Theorem provides a refinement of the choice of transcendental extensions so that certain ring extensions are integral extensions, not just algebraic extensions.

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