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Directed Reading Program

A Geometric Approach of Differential Forms

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A problem with parametric integration

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Proof.

if $\phi_1(x) = (x, \sqrt{1 - x^2})$ and $\phi_2(\theta) = (\cos(\theta), \sin(\theta))$

$\int_{-1}^{1} 1 - x^2 \, dx \neq \int_0^\pi \sin^2(\theta) \, d\theta$ for $\int y^2 \, dy$ over the top half of a circle
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\int_{-1}^{1} 1 - x^2 \, dx \neq \int_{0}^{\pi} \sin^2(\theta) \, d\theta
\]
for \( \int y^2 \, dy \) over the top half of a circle

Our problem stems from the fact that the points \( \phi_1(a_i) \) are not evenly spaced along the curve.
A calculated transformation

Using Riemann sums we can find an appropriate integral

\[ \sum_{i=1}^{n} F(a_i) \Delta a = \sum_{i=1}^{n} f(\phi_1(a_i))L_i \] as \( n \) goes to infinity
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\[ \lim_{\Delta a \to 0} F(a_i) = \lim_{\Delta a \to 0} \frac{f(\phi_1(a_i))L_i}{\Delta a} \] when boiled down has

\[ F(a) = f(\phi_1(a_i)) \left| \frac{d\phi_1}{da} \right| da \]

for lines in \( \mathbb{R}^2 \) and similarly for surfaces in \( \mathbb{R}^3 \)

\[ F(a) = f(\phi(a, b)) \text{Area} \left( \frac{d\phi}{da}, \frac{d\phi}{db} \right) da \, db \]
Forms

Definition (1-form)

A 1-form is simply a linear function denoted by $\omega$ which is fed a vector and:

- projects onto each coordinate axis, scaling each by some constant and adding the result
- projects onto some line and then multiplies by some constant

The vectors fed exist on the tangent space to a point denoted by $T_p \mathbb{R}^2$ for a line and $T_p \mathbb{R}^3$ for a surface
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$$\omega \wedge \nu(V_1, V_2) = \begin{vmatrix} \omega(V_1) & \omega(V_2) \\ \nu(V_1) & \nu(V_2) \end{vmatrix}$$
Multiplying 1-forms

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\end{vmatrix}
\]

**Definition**

for $\omega = adx + bdy + cdz$, $\langle \omega \rangle = \langle a, b, c \rangle$

It is possible to show through the linearity of forms two unique geometric and algebraic definitions
\[ \omega \wedge \upsilon = |\langle \omega \rangle| |\langle \upsilon \rangle| \bar{\omega} \wedge \bar{\upsilon} \]

Evaluating \( \omega \wedge \upsilon \) on the pair of vectors \((V_1, V_2)\) gives the area of the parallelogram spanned by \(V_1\) and \(V_2\) projected onto the plane containing the vectors \(\langle \omega \rangle\) and \(\langle \upsilon \rangle\), and multiplied by the area of the parallelogram spanned by \(\langle \omega \rangle\) and \(\langle \upsilon \rangle\)
\[ \omega \wedge \nu = |\langle \omega \rangle| |\langle \nu \rangle| \bar{\omega} \wedge \bar{\nu} \]

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\[ \omega \wedge \nu = c_1 dx \wedge dy + c_2 dx \wedge dz + c_3 dy \wedge dz \]

Every 2-form projects the parallelogram spanned by \( V_1 \) and \( V_2 \) onto each of the \( (2 \text{-dimensional}) \) coordinate planes, computes the resulting (signed) areas, multiplies each by some constant, and adds the results
Differential Forms

Forms that are differentiable on a given interval can be used more easily for our integral previously on the part which involved $\text{Area}(\frac{d\phi}{da}, \frac{d\phi}{db})$
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Example

for the differential 2 form acting on two vector fields $V_1 = \langle 2y, 0, -x \rangle$ and $V_2 = \langle z, 1, xy \rangle$ with

$\omega = x^2y \, dx \wedge dy - xz \, dy \wedge dz$

$\omega(V_1, V_2) = x^2y \begin{vmatrix} 2y & z & 0 \\ 0 & 1 & -x \end{vmatrix} - xz \begin{vmatrix} 0 & 1 \\ -x & xy \end{vmatrix} = 2x^2y^2 - x^2z$
Integrating Forms

Through a similar proof as with the integral of a function we associate the integral of a 2-form in $\mathbb{R}^3$ through equivalent Riemann sums

$$\sum_i \sum_j f(x_i, y_j) dx \wedge dy(V_{i,j}^1, V_{i,j}^2) = \sum_i \sum_j \omega_\phi(x_i, y_j)(V_{i,j}^1, V_{i,j}^2)$$
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Through extensive calculations becomes:

$$\int_M \omega = \int_R \omega_\phi(x, y) \left( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right) dx \wedge dy$$
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$$\sum_i \sum_j f(x_i, y_j) dx \wedge dy(V_{i,j}^1, V_{i,j}^2) = \sum_i \sum_j \omega(\phi(x_i, y_j)) (V_{i,j}^1, V_{i,j}^2)$$

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The orientation of the parametrization is crucial and can switch based on the choice of $V_{i,j}^1$ and $-V_{i,j}^1$
Differentiating Forms

Since the variation of forms depends on the direction from an arbitrary point \( p \), the derivative of a 1 form can only be taken as a 2-form with a chosen vector \( W \).

\[
d\omega(V) = \nabla \omega(V) \cdot W
\]

Example for \( \omega = ydx - x^2dy \) with \( V = \langle 1, 2 \rangle \) and \( W = \langle 2, 3 \rangle \):

\[
d\omega = \langle -4x, 1 \rangle \cdot \langle 2, 3 \rangle = -8x + 3
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An Approach to Stokes Theorem

an n-cell is defined as the image of a differentiable map and denoted by \( \sigma \) where as a chain is a formal linear combination of n-cells where

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the boundary of a n-cell is denoted by $\partial\sigma$ and is formulated

$$
\partial\sigma = \sum_{i=1}^{n} (-1)^{i+1} (\phi(x_1,\ldots,x_{i-1},1) - \phi(x_1,\ldots,x_{i-1},0))
$$
An Approach to Stokes Theorem

**Generalized Stokes Theorem:**

\[ \int_{\partial \sigma} \omega = \int_{\sigma} d\omega \]
An Approach to Stokes Theorem

**Generalized Stokes Theorem:**
\[
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\]

**Actual Stokes Theorem:**
\[
\int_{C} \vec{F} \cdot d\vec{r} = \int_{S} \text{Curl} \vec{F} \cdot n d\vec{S}
\]
THANK YOU!