ZEV KAPLOWITZ

Directed Reading Program

A Geometric Approach of Differential Forms

University of Maryland

A problem with parametric integration

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Our problem stems from the fact that the points $\phi_1(a_i)$ are not evenly spaced along the curve

A calculated transformation

Using Riemann sums we can find an appropriate integral

$$\sum_{i=1}^{n} F(a_i)\Delta a = \sum_{i=1}^{n} f(\phi_1(a_i))L_i \text{ as n goes to infinity}$$

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$\lim_{\Delta a \to 0} F(a_i) = \lim_{\Delta a \to 0} \frac{f(\phi_1(a_i))L_i}{\Delta a}$ when boiled down has

$$F(a) = f(\phi_1(a_i)) \left| \frac{d\phi_1}{da} \right| da$$

for lines in $\mathbb{R}^2$and similarly for surfaces in $\mathbb{R}^3$

$$F(a) = f(\phi(a, b)) Area(\frac{d\phi}{da}, \frac{d\phi}{db}) da\, db$$

Forms

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The vectors fed exist on on the tangent space to a point denoted by $T_p\mathbb{R}^2$ for a line and $T_p\mathbb{R}^3$ for a surface

Multiplying 1-forms

The multiplication of two one forms, say $\omega(V_1)$ and $\upsilon(V_2)$, is considered a 2-form and denoted by $\omega \wedge \upsilon(V_1, V_2)$ and is evaluated in the following way:

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Definition

for $\omega = adx + bdy + cdz, \quad \langle\omega\rangle = \langle a, b, c\rangle$

It is possible to show through the linearity of forms two unique geometric and algebraic definitions

$$\omega \wedge \upsilon = |\langle\omega\rangle|\,|\langle\upsilon\rangle|\,\bar{\omega} \wedge \bar{\upsilon}$$

Evaluating $\omega \wedge \upsilon$ on the pair of vectors (V_1, V_2) gives the area of the parallelogram spanned by V_1 and V_2 projected onto the plane containing the vectors $\langle\omega\rangle$ and $\langle\upsilon\rangle$, and multiplied by the area of the parallelogram spanned by $\langle\omega\rangle$ and $\langle\upsilon\rangle$

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$$\omega \wedge \upsilon = c_1 dx \wedge dy + c_2 dx \wedge dz + c_3 dy \wedge dz$$

Every 2-form projects the parallelogram spanned by V_1 and V_2 onto each of the (2-dimensional) coordinate planes, computes the resulting (signed) areas, multiplies each by some constant, and adds the results

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Forms that are differentiable on a given interval can be used more easily for our integral previously on the part which involved $Area(\frac{d\phi}{da}, \frac{d\phi}{db})$

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Example

for the differential 2 form acting on two vector fields $V_1 = \langle 2y, 0, -x \rangle$ and $V_2 = \langle z, 1, xy \rangle$ with

$$\omega = x^2 y dx \wedge dy - xz dy \wedge dz$$

$$\omega(V_1, V_2) = x^2 y \begin{vmatrix} 2y & z \\ 0 & 1 \end{vmatrix} - xz \begin{vmatrix} 0 & 1 \\ -x & xy \end{vmatrix} = 2x^2y^2 - x^2 z$$

Integrating Forms

Through a similar proof as with the integral of a function we associate the integral of a 2-form in $\mathbb{R}^3$ through equivalent Riemann sums

$$\sum_i \sum_j f(x_i, y_j) dx \wedge dy(V_{i,j}^1, V_{i,j}^2) = \sum_i \sum_j \omega_{\phi(x_i, y_j)}(\mathbb{V}_{i,j}^1, \mathbb{V}_{i,j}^2)$$

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Through extensive calculations becomes:

$$\int_M \omega = \int_R \omega_{\phi(x,y)}(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}) dx \wedge dy$$

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The orientation of the parametrization is crucial and can switch based on the choice of $V_{i,j}^1$ and $-V_{i,j}^1$

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Example

for $\omega = ydx - x^2dy$ with $V = \langle 1, 2 \rangle$ and $W = \langle 2, 3 \rangle$

$$dw = \langle -4x, 1 \rangle \cdot \langle 2, 3 \rangle = -8x + 3$$

An Approach to Stokes Theorem

an n-cell is defined as the image of a differentiable map and denoted by σ where as a chain is a formal linear combination of n-cells where

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the boundary of a n-cell is denoted by $\partial\sigma$ and is formulated

$$\partial\sigma = \sum_{i=1}^{n} (-1)^{i+1} (\phi_{(x_1,\ldots,x_{i-1},1)} - \phi_{(x_1,\ldots,x_{i-1},0)})$$

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Actual Stokes Theorem:

$$\int_{C} \vec{F} \cdot d\vec{r} = \iint_{S} Curl\,\vec{F} \cdot n\,d\vec{S}$$

THANK YOU!