The Brouwer Fixed Point Theorem

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Overview

As the story has it, L.E.J. Brouwer, a Dutch mathematician and philosopher, observed that as one stirs a cup of coffee, to dissolve a lump of sugar, it appears that there is always at least one point without motion. The mathematical formulation of this statement is that any continuous map from the closed disk to itself, $f : D^2 \to D^2$, has a fixed point. Our goal is to give a proof of this theorem.

The presentation is as follows:

- What is a Group?
- What is a topological space?
- The Fundamental group.
- Brouwer's Fixed point theorem.

What is a group?

A group is a set G together with a binary operation $\cdot : G \times G \to G$ that combines two elements a, b of G to form an element $a \cdot b$ such that the thee axioms are satisfied:

1. (associativity) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$,
2. (identity element) there is an element $e \in G$ such that $a \cdot e = e \cdot a = a$, for all elements a in G,
3. (inverse element) for each $a \in G$ there is an element $b \in G$, usually denoted by a^{-1}, such that $a \cdot b = b \cdot a = e$.

What is a group?

The following are some examples of groups;

- $(\mathbb{Z}, +)$ - integers under addition,
- $(\mathbb{Q}, +)$ - rational numbers under addition,
- $(\mathbb{R} \setminus 0, \cdot)$ - real numbers without 0 under multiplications.

The following are not groups:

- $(\mathbb{Z}, -)$ integers under subtraction is not a group because associativity is not satisfied,
- $(\mathbb{Z}, \cdot)$ integers under multiplication is not a group because there are no inverses,
- $(\mathbb{N}, +)$ natural numbers under addition is not a group because there are no inverses.

Topological spaces

Topological spaces are sets together with a notion of "distance", in the sense that, we would like to tell when two points are close or not. One way to do this is to endow the set with a metric, however, a more general definition of a topological space is as follows:

Definition

A set X, together with a collection of its subsets $\mathcal{T}$, is called a topological space if the following axioms hold

1. The empty set $\emptyset$, and the whole space X, is in T,
2. the intersection of a finite number of sets in T is also in T,
3. the union of an arbitrary number of sets n $\mathcal{T}$ is also in $\mathcal{T}$.

Continuous maps

Now that we have a notion of "distance" on a set, we would like to define continuous functions between two topological spaces $f : X \to Y$ such that, if two points x, y are "close" in X then so are $f(x), f(y)$ in Y. More specifically,

Definition

We say that a map $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is a continuous map or continuous function if for any sequence of points $\{x_n\}_n$ in X converging to a point $x \in X$, the sequence $\{f(x_n)\}_n$ converges to $f(x)$ in Y.

Paths and loops

A *path* on a topological space X is simply a continuous map γ from $[0,1]$ to X, usually denoted by $\gamma : [0,1] \to X$. A path is called a *loop* is the start point and the end point coincide, i.e., $\gamma(0) = \gamma(1)$.

Note that a constant map $\gamma : [0,1] \to X$, that is $\gamma(t) = x$, for all $t \in [0,1]$, for some fixed $x \in X$ is also a loop, and it's called the *constant loop*. A constant loop is basically an interval mapped to a single point x.

Homotopy

As we'll soon see, for any topological space $(X, \mathcal{T})$, we would like to form "the group of all loops starting at a fixed point". However, any slight deformation of a loop at any point would result in a different loop, making the group to large to handle. That's why we would like to consider all the loops that can be deformed to each other as one element of the group. We consider two paths homotopic if we can deform one to another.

Homotopy

Definition

We say two paths $\gamma_0, \gamma_1 : [0,1] \to X$, such that $\gamma_0(0) = \gamma_1(0) = x_0$, and $\gamma_0(1) = \gamma_1(1) = x_1$, are *homotopic* is there is a continuous

$$H : [0,1] \times [0,1] \to X,$$

such that $H(0,u) = \gamma_0(u)$, $H(1,u) = \gamma_1(u)$, $H(t,0) = x_0$, and $H(t,1) = x_1$.

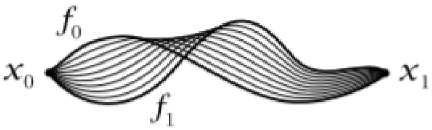


Figure: Homotopy of paths

Fundamental group

Having the notion of homotopy we can now define the *fundamental group* of a topological space. As we said, we would like to form the group of all loops starting at some fixed point $x_0 \in X$, such that all the loops that are deformations of each other form just one element.

Let X be a topological space, and x_0 a point on X. We define the *fundamental group with basepoint* x_0

$$\pi_1(X, x_0) = \{\text{all loops } \gamma \text{ based on } x_0\} / \text{homotopy}$$

Note that for the spaces we are interested in it doesn't matter what we choose as basepoint, and hence we may sometimes write $\pi_1(X)$ instead of $\pi_1(X, x_0)$.

Examples of Fundamental groups

- For the two dimensional sphere S^2, one can see that any loop can be deformed to a constant loop, thus there is only one element in the fundamental group and hence $\pi_1(S^2) = 0$.

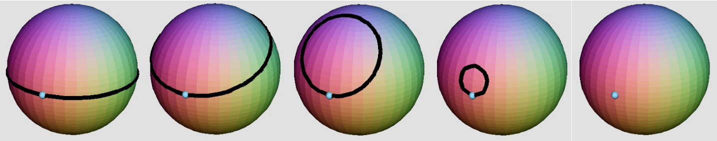


Figure: Fundamental group of sphere

- Similarly, if D^2 denotes the two dimensional disk, all loops can be deformed to the constant loop and hence

$$\pi_1(D^2) = 0.$$

Examples of Fundamental groups

- For the circle S^1, the existence of a "hole" makes it impossible to deform to the constant loop any loop that has done a full turn around the circle without breaking it. Moreover, loops that turn twice around the circle are different than loops that only turn only once, and also direction matters. As a result, we get one loop for each integer, the sign of which determines the direction, and its absolute value determines the number of turns around the circle. As a result,

$$\pi_1(S^1) = \mathbb{Z}.$$

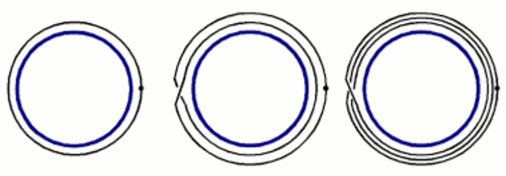


Figure: Fundamental group of circle

Examples of Fundamental groups

- Similarly, for a torus T^2, that is the surface of a donut, we have

$$\pi_1(T^2) = \mathbb{Z}^2.$$

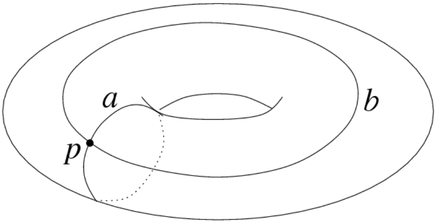


Figure: Fundamental group of torus.

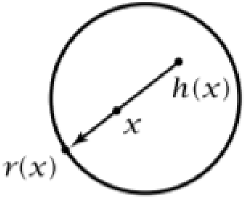
Brouwer's fixed point theorem

We are now ready to state and sketch the proof of our main theorem.

Theorem (Brouwer fixed point theorem)

A continuous map $h : D^2 \to D^2$ has a fixed point. That is, there is $x \in X$ such that $h(x) = x$.

Proof. We'll prove it by contradiction. Suppose we could find a continuous map $h : D^2 \to D^2$ without any fixed point. Then, we can construct a continuous map $r : D^2 \to S^1$ as follows; Let $x \in D^2$, since $h(x) \neq x$, we can form a line from x to $h(x)$. Let $r(x)$ be the point on the intersection of that line and S^1 that is closest to x, as in the figure.



Note that if we start with a point $x \in S^1 \subset D^2$ then $r(x) = x$, and hence if $i : S^1 \to D^2$ the inclusion, then $r \circ i = \text{id}$.

$$\begin{tikzcd} S^1 \arrow[r, "i"] \arrow[rr, bend right, "\text{Id}"'] & D^2 \arrow[r, "r"] & S^1 \end{tikzcd}$$

Figure: Diagram 1

But, continuous maps on spaces induce maps on loops, so we get a corresponding

$$\begin{tikzcd} \pi_1(S^1) \arrow[r, "i_*"] \arrow[rr, bend right, "\text{Id}"'] & \pi_1(D^2) \arrow[r, "r_*"] & \pi^1(S^2) \end{tikzcd}$$

Figure: Diagram 2

Brouwer's fixed point theorem

However, we've computed $\pi_1(S^1) = \mathbb{Z}$, and $\pi_1(D^2) = 0$, and hence we have:

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "i_*"] \arrow[rr, bend right, "\mathrm{Id}"'] & 0 \arrow[r, "r_*"] & \mathbb{Z} \end{tikzcd}$$

Figure: Diagram 3

This means that $1 \in \mathbb{Z}$ should map to 0 and then 0 should map to 1 because the composition of the two maps is the identity. However, the 0 element of a group always maps to 0 under a group homomorphism, giving us a contradiction. $\square$