Bayesian Games

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Structure of a Game

- Normal Form Games:
  - Set of finite players: $N = \{1,2,\ldots,n\}$
  - Collection of sets of pure strategies: $\{S_1, S_2, \ldots, S_n\}$
  - A set of payoff functions: $\{v_1, v_2, \ldots v_n\}$ $v_i: S_1 \times S_2 \times \cdots \times S_n \rightarrow \mathbb{R}$
- Best Response:
  - $s_i \in S_i$ is a best response to the oppents stratgey s_{-i} if $v_i(s_i, s_{-i}) \geq v_i(s'_i, s_{-i})$ $\forall s'_i \in S_i$
- Strictly Dominant Strategy:
  - s_i is a strictly dominant strategy if $v_i(s_i, s_{-i}) > v_i(s'_i, s_{-i})$ $\forall s'_i \in S_i, s'_i \neq s_i, and$ $\forall s_{-i} \in S_{-i}$

Defining Nash Equilibrium

- The strategy profile $s^* = (s_1^*, s_2^*, \ldots, s_n^*) \in S$ is a Nash Equilibirum if $v_i(s_i^*, s^*_{-i}) \geq v_i(s'_i, s^*_{-i})\ \forall s'_i \in S_i, \forall i$
- Requirements for Nash Equilibrium:
  1. Each player is playing a best response to their beliefs
  2. The beliefs(s_{-i}) about their opponents are correct
- Example:

	$Quiet$	$Tell$
$Quiet$	$-2, -2$	$-5, -1$
$Tell$	$-1, -5$	$\mathbf{-4}, \mathbf{-4}$

  - It is clear that Tell is a Strictly Dominant strategy for both players. This is called a strict dominant strategy equilibrium.

Setting up the Cournot Duopoly

- Normal Form setup:
  - Players: $N = \{1,2\}$
  - Strategy sets: $S_i = [0, \infty)\ for\ i \in \{1,2\}. Firms\ choose\ quantity\ q_i \in S_i$
  - Payoffs: $For\ i, j\ \in \{1,2\}, i \neq j, v_i(q_i, q_j) = \left(a - b(q_i + q_j)\right) q_i - cq_i$
    - Note: Sale price of good = $a - bq$ where $q = q_i + q_j$ is total quantity. Cost for firm i = cq_i
- We are trying to maximize the payoff function with the belief that the opponent is choosing quantity q_j

Cournout Duopoly Nash Equilibrium

1. Calculate Best Response of firm i given firm j produces q_j
   - $\frac{\partial}{\partial q_i} v_i(q_i, q_j) = \frac{\partial}{\partial q_i}\left(a - b(q_i + q_j)\right) q_i - cq_i = a - 2bq_i - bq_j - c = 0$
   - $q_i = BR_i(q_j) = \frac{a - bq_j - c}{2b}$

2. We can now find the Nash Equilibrium
   - $q_1 = \frac{a - bq_2 - c}{2b}, q_2 = \frac{a - bq_1 - c}{2b}$
   - Given each firms will play their best response, we can solve the system for the Nash Equilibrium
   - Nash Equilibrium: $q_i^* = \frac{a - c}{3}$

Graphical Representation of Cournot Duopoly

$$q_i = BR_i(q_j) = \frac{a - bq_j - c}{2b}$$

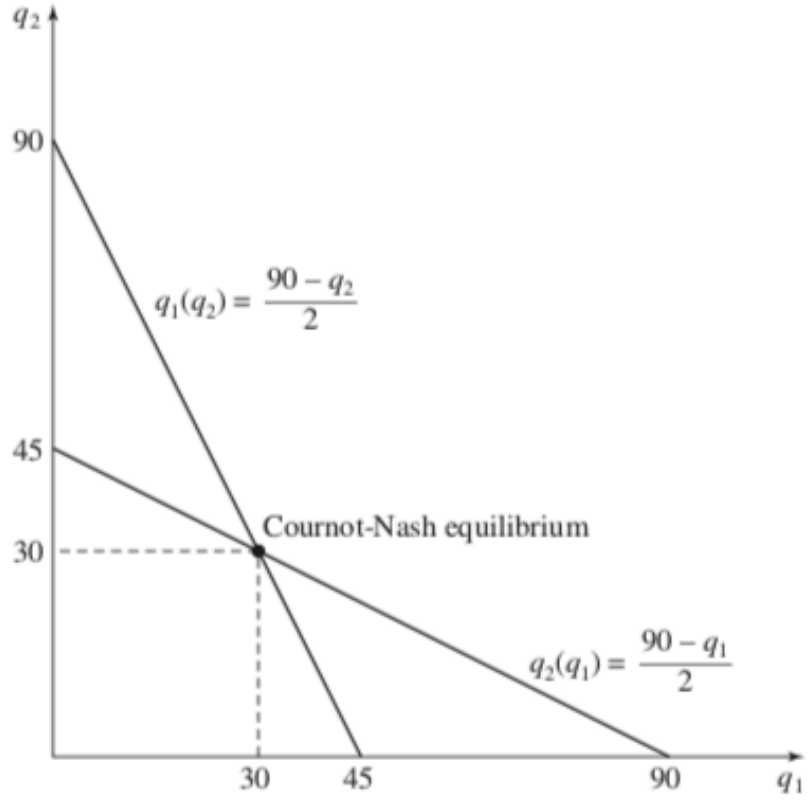

FIGURE 5.2 Cournot duopoly game: best-response functions and Nash equilibrium.

Note: a= 100, b=1, c = 10.

Structure of Bayesian Games

- Normal Form Bayesian Game:
  - Set of finite players: $N = \{1,2,\dots,n\}$
  - Collection of sets of pure strategies: $S = \{S_1, S_2, \dots, S_n\}$
  - A type space for player i: $\Theta_i = \{\theta_{i1}, \theta_{i2}, \dots, \theta_{ik_i}\}$
  - A type dependent payoff function: $\{v_1, v_2, \dots v_n\}$ $v_i: S \times \Theta_i \to \mathbb{R}$
  - A set of beliefs of opponents types: $\phi_i(\theta_{-i}|\theta_i)$
    - Conditional Distribution of opponents types given Player i knows their own type
- Player's own type is private information
- Strategy space, payoff functions, possible types, and beliefs are common knowledge to all players

Setting up Bayesian Cournot Duopoly

- Normal Form Setup:
  - Players: $N = \{1,2\}$
  - Strategy sets: $S_i = [0, \infty)\ for\ i \in \{1,2\}. Firms\ choose\ quantity\ q_i \in S_i$
  - Type space for firm i: $\Theta_i = \{C_L, C_H\}$
    - C_L, C_H represent low cost of production and high cost of production, respectively
  - Payoffs: $For\ i, j\ \in \{1,2\}, i \neq j, v_i(q_i, q_j) = \left(a - b(q_i + q_j)\right) q_i - c_i q_i$
  - Beliefs for firm i: $\Pr\{c_j = C_L\} = \mu, \Pr\{c_j = C_H\} = 1 - \mu$
    - Values of μ and $1 - u$ are common knowledge
    - Own type c_i is private information

- Bayesian Nash Equilibrium: $(q_{1L}^*, q_{1H}^*, q_{2L}^*, q_{2H}^*)$

Bayesian Cournot Duopoly Nash Equilibrium

- Solve for best responses for $q_{1L}^*, q_{1H}^*, q_{2L}^*, q_{2H}^*$
  - $v_1(q_{1L}, q_2^*) = \mu\big[\big(a - b(q_{1L} + q_{2L})\big)q_{1L} - C_L q_{1L}\big] + (1-\mu)\big[\big(a - b(q_{1L} + q_{2H})\big)q_{1L} - C_L q_{1L}\big]$
  - $\frac{\partial}{\partial q_{1L}} v_1(q_{1L}, q_2^*) = 0 = a - 2bq_{1L} - \mu q_{2L}^* - (1-\mu)q_{2H}^* - C_L$
  - Implies: $q_{1L}^* = \frac{a - \mu q_{2L}^* - (1-\mu)q_{2H}^* - C_L}{2b}$
- $\begin{bmatrix} q_{1L}^* \\ q_{1H}^* \\ q_{2L}^* \\ q_{2H}^* \end{bmatrix} \begin{bmatrix} 2b & 0 & \mu & 1-\mu \\ 0 & 2b & \mu & 1-\mu \\ \mu & 1-\mu & 2b & 0 \\ \mu & 1-\mu & 0 & 2b \end{bmatrix} = \begin{bmatrix} a - C_L \\ a - C_H \\ a - C_L \\ a - C_H \end{bmatrix}$

Structure of Auctions

- English Auction:
  - Price of the good goes up as long as someone is willing to bid higher. Once bid is not challenged, the auction ends and bidder pays price.
  - Optimal Strategy: Bid exactly how much you value the good
- Dutch Auction:
  - Price starts high and auctioneer lowers until a bidder shouts "buy". Auction ends and bidder pays price at which they called out.
- Distribution of Types:
  - Value of good for player i: $\theta_i \in \left[\underline{\theta_i}, \overline{\theta_i}\right], \underline{\theta_i} \geq 0$
  - θ_i is drawn according to CDF $F_i(\cdot)$ $such\ that F_i(\theta') = \Pr\{\theta_i \leq \theta'\}$
  - Bid Function: $b_i = s_i(\theta_i): \theta_i \rightarrow \mathbb{R}$

Dominant Bid Strategy for Dutch Auctions

- Assumptions:
  - All players have the same bid function s
  - Bid function is invertible i.e. $s_j^{-1}(b_j) = \theta_j$
  - CDF F is uniform on [0,1] for all players
  - All players have values distributed on the same interval $[\underline{\theta}, \overline{\theta}]$
- Finding Payoff function for player i:
  - $\Pr\{b_j < b_i\} = \Pr\{s(\theta_j) < b_i\} = \Pr\{\theta_j < s^{-1}(b_i)\} = F(s^{-1}(b_i))$
  - Expected payoff = $\Pr\{all\ other\ bids < b_i\}\,(\theta_i - b_i)$
  - Expected payoff =$[F(s^{-1}(b_i))]^{n-1}(\theta_i - b_i)$
    - Objective is to maximize bid b_i

Dominant Bid Strategy for Dutch Auctions cont.

- First Order Condition:
  - $\frac{\partial}{\partial b_i}[F(s^{-1}(b_i))]^{n-1}(\theta_i - b_i) = 0$
  - $-[F(s^1(b_i))]^{n-1} + (n-1)[F(s^{-1}(b_i))]^{n-2} \cdot f(s^{-1}(b_i)) \cdot \frac{ds^{-1}(b_i)}{db_i}(\theta_i - b_i) = 0$
- Solving FOC:
  - Looking for $s^*(\theta) = b$ that solves the FOC
  - We eventually find, $[F(\theta)]^{n-1}s(\theta) = \int_{\underline{\theta}}^{\theta_i}(n-1)[F(x)]^{n-2}f(x)xdx$
  - $s(\theta) = \theta - \frac{\int_0^{\theta_i} x^{n-1}dx}{\theta_i^{n-1}} = \theta - \frac{\theta^n}{n\theta^{n-1}} = \theta_i(\frac{n-1}{n})$ is the optimal bid for player i

Where to go Next

- N-player Cournot game with incomplete information
- Changing the assumptions for Dutch Auction
  - Change the distribution of types
  - No standard interval or bet function for all players
- Signaling games and value of information
  - Adverse Selection games

References

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