Skolem's "Paradox"

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DRP Talks, May 2014

Intro

- Skolem's Paradox: theorem of set theory.
- "Not so much a paradox in terms of outright contradiction, but rather a kind of anomaly" - Stephen Kleene, American Logician.

- Logical Symbols: $\wedge$, $\vee$, $\neg$, $\forall$, $\exists$, $\rightarrow$, $\leftrightarrow$, $=$,...
- Variables: x_1, x_2, x_3...
- Function/Constant/Relation Symbols: f_1, R_1, f_2, R_2,...

Example

The language of a ring with unity, besides having logical symbols, has 0, 1, $\bullet$, $+$.

Sentences and Formulas

Predicate Logic

- Sentence: A string of symbols with a truth value.
- Formula: Would be a sentence if free variables are instantiated or quantified.

Example

Let $\phi(x)$ be the formula "$x < 0$". We say that $\phi(x)$ is a formula with free variable x. Then, $\exists x \phi(x)$ says "$\exists x(x < 0)$" and $\phi(0)$ says "$0 < 0$", both sentences corresponding to $\phi(x)$.

Axioms of ZFC

Some Examples

- ZFC: Axiomatic Treatment of Set Theory
- All variables represent objects which we call 'sets', and our axioms are in terms of the relation symbol $\in$.
- Extensionality: A set is determined by its members: $\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y)$
- Comprehension: For each formula $\phi(y)$ with only y occurring as a free variable, for any set x, $\{z \in x : \phi(z)\}$ exists.
- Pairing: $\forall x \forall y \exists z (x \in z \wedge y \in z)$.

Example

Given x and y, Pairing guarantees a z such that $x \in z, y \in z$. By Comprehension, $\{x, y\} = \{v \in z : v = x \vee v = y\}$ exists, and is unique by Extensionality.

Ordinals

- A formal way of thinking of Natural numbers and beyond.

Definition

The following is a definition for finite ordinals:

1. $0 = \{\}$, the empty set, also denoted $\emptyset$, is an ordinal
2. If α is an ordinal, $S(\alpha) = \alpha \cup \{\alpha\}$ is also an ordinal.

Example

$1 = \{0\} = \{\{\}\}$
$2 = \{0, 1\} = \{\{\}, \{\{\}\}\}$
$3 = \{0, 1, 2\} = \{0, 1, \{0, 1\}\}$
$n = \{1, 2, 3, ..., n - 1\}$

- Infinity: $\exists x(0 \in x \wedge \forall y \in x(S(y) \in x))$

Definition

The minimal set satisfying the Axiom of Infinity is called ω.

Remark

ω is the set of natural numbers.

Definition

A set S is said to be **countable** if there exists $f : \omega \to S$ such that f is onto.

- Power Set: For each set x, there is a set containing every subset of x.

Definition

$\mathcal{P}(x) = \{z : z \subset x\}$ which is a subset of the set guaranteed by the Power Set Axiom.

Theorem

For all x, there is no function from x onto $\mathcal{P}(x)$.

Corollary

There exists an uncountable set, namely, $\mathcal{P}(\omega)$.

- Given a set of symbols $\mathcal{L}$, the pair (A, V) is a **structure** for $\mathcal{L}$ if A is a non-empty set and V consists of definitions of the symbols in $\mathcal{L}$.
- A structure for some set of symbols $\mathcal{L}$, (A, V) is a **model** for a set of axioms Q, *forthesymbolsof* L*ifeverystatementin*Q*istruein*(A,V).

Example

Let $\mathcal{L} = \{0, 1, +, \times\}$. If V contains the standard definitions for 1, 0, $+$, $\times$, then $(\mathbb{Z}, V)$ is a structure for $\mathcal{L}$. If Q contains the axioms for a ring with unity, $(\mathbb{Z}, V)$ is a model of Q.

- (B, W) is a **substructure** of (A, V) if $B \subseteq A$ and W contains the definitions in V restricted to elements of B. We denote this by $(B, W) \subseteq (A, V)$.
- (B, W) is an **elementary substructure** of (A, V) if $(B, W) \subseteq (A, V)$ and for each sentence ϕ referencing only elements of B, ϕ is true in (A, V) if and only if ϕ is true in (B, W). Then, we write $(B, W) \preceq (A, V)$.

Example

For the standard interpretation of $\mathcal{L} = \{0, 1, +, \times\}$, $\mathbb{Q} \subseteq \mathbb{R}$. However, $\mathbb{Q} \not\preceq \mathbb{R}$ since $\exists x(x^2 = 2)$ is true in $\mathbb{R}$ but not in $\mathbb{Q}$.

Theorem (Lowenheim-Skolem)

Every structure has countable elementary substructure.

Example

The set of real algebraic numbers, $\overline{\mathbb{Q}} \backslash \mathbb{C}$, is a countable elementary substructure of $\mathbb{R}$.

Corollary

If ZFC is consistent, it has a countable model.

Skolem's Paradox

There exists a countable model containing an uncountable set.

How?

- This uncountable set is $\mathcal{P}(\omega)$, in particular.

Definition

$\mathcal{P}(\omega) = \{z : z \subset \omega\}$

Clarification

In a model of ZFC, (A, V), $\mathcal{P}^{\mathbf{A}}(\omega) = \{z \in A : z \subset \omega\}$

- Since A is countable and $\mathcal{P}^A(\omega) \subseteq A$, $\mathcal{P}^A(\omega)$ must be countable

How?

Definition

A set S is said to be countable if there exists $f : \omega \to S$ such that f is onto.

Clarification

A set S is said to be countable **in a model of ZFC**, $\boldsymbol{(A, V)}$, if there exists **in** $\boldsymbol{A}$ $f : \omega \to S$ such that f is onto

- So $\mathcal{P}^A(\omega)$ can still be uncountable in (A, V) if none of the functions which map ω onto $\mathcal{P}^A(\omega)$ are in A.
- In fact, the pairing axiom guarantees that each element of any function mapping ω onto $\mathcal{P}^A(\omega)$ are in A. However, ZFC provides no way of proving that their collection exists.

Outlook

- Axiomatizing doesn't always do what we want it to
- Lowenheim Skolem theorem tells us that this will be unavoidable