Classification of Semisimple Lie Algebras

Kyle Reese

December 2019

What is a Lie Group?

What is a Lie Group?

Definition
A Lie Group G is a group that is also a differentiable manifold such that its group operations are smooth.

What is a Lie Group?

Definition

A Lie Group G is a group that is also a differentiable manifold such that its group operations are smooth.

Example

The set of matrices $GL(n, \mathbb{C})$ is a Lie Group under matrix multiplication.

What is a Lie Group?

Definition

A Lie Group G is a group that is also a differentiable manifold such that its group operations are smooth.

Example

The set of matrices $GL(n, \mathbb{C})$ is a Lie Group under matrix multiplication.
So is $SL(n, \mathbb{C})$

What is a Lie Group?

Definition

A Lie Group G is a group that is also a differentiable manifold such that its group operations are smooth.

Example

The set of matrices $GL(n, \mathbb{C})$ is a Lie Group under matrix multiplication.
So is $SL(n, \mathbb{C})$, and $U(n)$

What is a Lie Group?

Definition

A Lie Group G is a group that is also a differentiable manifold such that its group operations are smooth.

Example

The set of matrices $GL(n, \mathbb{C})$ is a Lie Group under matrix multiplication.
So is $SL(n, \mathbb{C})$, and $U(n)$, $SO(n)$

What is a Lie Group?

Definition

A Lie Group G is a group that is also a differentiable manifold such that its group operations are smooth.

Example

The set of matrices $GL(n, \mathbb{C})$ is a Lie Group under matrix multiplication.
So is $SL(n, \mathbb{C})$, and $U(n)$, $SO(n)$, and more.

What is a Lie Group?

Definition

A Lie Group G is a group that is also a differentiable manifold such that its group operations are smooth.

Example

The set of matrices $GL(n, \mathbb{C})$ is a Lie Group under matrix multiplication.
So is $SL(n, \mathbb{C})$, and $U(n)$, $SO(n)$, and more.

Definition

A Matrix Lie Group is a **closed** subgroup $G \leq GL(n, \mathbb{C})$. That is, whenever $\{A_n\} \subseteq G$ converges to A, then either $A \in G$ or $A \notin GL(n, \mathbb{C})$.

What is a Lie Group?

Definition

A Lie Group G is a group that is also a differentiable manifold such that its group operations are smooth.

Example

The set of matrices $GL(n, \mathbb{C})$ is a Lie Group under matrix multiplication.
So is $SL(n, \mathbb{C})$, and $U(n)$, $SO(n)$, and more.

Definition

A Matrix Lie Group is a **closed** subgroup $G \leq GL(n, \mathbb{C})$. That is, whenever $\{A_n\} \subseteq G$ converges to A, then either $A \in G$ or $A \notin GL(n, \mathbb{C})$.

For example: $SL(n, \mathbb{C})$ is a Matrix Lie Group because it is a subgroup of $GL(n, \mathbb{C})$, and if $\{A_n\} \subseteq SL(n, \mathbb{C})$ converges to A, then $A \in SL(n, \mathbb{C})$ because each A_n has determinant one and the determinant function is continuous.

Tangent Spaces and Lie Algebras

Tangent Spaces and Lie Algebras

Definition

A Lie Algebra $\mathfrak{g}$ is a vector space along with a map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that is bilinear, skew symmetric, and satisfies the Jacobi Identity:

Tangent Spaces and Lie Algebras

Definition

A Lie Algebra $\mathfrak{g}$ is a vector space along with a map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that is bilinear, skew symmetric, and satisfies the Jacobi Identity:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

Tangent Spaces and Lie Algebras

Definition

A Lie Algebra $\mathfrak{g}$ is a vector space along with a map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that is bilinear, skew symmetric, and satisfies the Jacobi Identity:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

Two elements $X, Y \in \mathfrak{g}$ are said to commute if $[X, Y] = 0$.

Tangent Spaces and Lie Algebras

Definition

A Lie Algebra $\mathfrak{g}$ is a vector space along with a map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that is bilinear, skew symmetric, and satisfies the Jacobi Identity:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

Two elements $X, Y \in \mathfrak{g}$ are said to commute if $[X,Y] = 0$.

- Given a differentiable manifold M and a point $p \in M$, the set of tangent vectors at p is denoted $T_p(M)$.

Tangent Spaces and Lie Algebras

Definition

A Lie Algebra $\mathfrak{g}$ is a vector space along with a map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that is bilinear, skew symmetric, and satisfies the Jacobi Identity:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

Two elements $X, Y \in \mathfrak{g}$ are said to commute if $[X, Y] = 0$.

- Given a differentiable manifold M and a point $p \in M$, the set of tangent vectors at p is denoted $T_p(M)$.
- Every Lie Group G has an associated Lie Algebra $\mathfrak{g} = T_0 G$, where $0 \in G$ is the identity element.

Tangent Spaces and Lie Algebras

Definition

A Lie Algebra $\mathfrak{g}$ is a vector space along with a map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that is bilinear, skew symmetric, and satisfies the Jacobi Identity:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

Two elements $X, Y \in \mathfrak{g}$ are said to commute if $[X, Y] = 0$.

- Given a differentiable manifold M and a point $p \in M$, the set of tangent vectors at p is denoted $T_p(M)$.
- Every Lie Group G has an associated Lie Algebra $\mathfrak{g} = T_0 G$, where $0 \in G$ is the identity element.

Example

$\mathbb{R}^3$ with $[x, y] = x \times y$ is also a Lie Algebra.

Tangent Spaces and Lie Algebras

Definition

A Lie Algebra $\mathfrak{g}$ is a vector space along with a map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that is bilinear, skew symmetric, and satisfies the Jacobi Identity:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

Two elements $X, Y \in \mathfrak{g}$ are said to commute if $[X,Y] = 0$.

- Given a differentiable manifold M and a point $p \in M$, the set of tangent vectors at p is denoted $T_p(M)$.
- Every Lie Group G has an associated Lie Algebra $\mathfrak{g} = T_0 G$, where $0 \in G$ is the identity element.

Example

$\mathbb{R}^3$ with $[x,y] = x \times y$ is also a Lie Algebra. $gl(V)$, the set of linear maps from V to itself, is a Lie Algebra with bracket $[x,y] = xy - yx$.

Matrix Lie Algebras

Recall the matrix exponential map:

Matrix Lie Algebras

Recall the matrix exponential map:

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

Matrix Lie Algebras

Recall the matrix exponential map:

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

It can be shown that this above mapping converges for any complex-valued A, and is in fact continuous.

Matrix Lie Algebras

Recall the matrix exponential map:

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

It can be shown that this above mapping converges for any complex-valued A, and is in fact continuous.
In the case that G is a Matrix Lie Group, the Lie Algebra of G can be computed more practically as the set of complex matrices X such that $e^{tX} \in G$ for every real t.

Example: Computing the Lie Algebra of $SL(n, \mathbb{C})$

Example: Computing the Lie Algebra of $SL(n, \mathbb{C})$

- We seek matrices such that $e^{tX} \in SL(n, \mathbb{C})$

Example: Computing the Lie Algebra of $SL(n, \mathbb{C})$

- We seek matrices such that $e^{tX} \in SL(n, \mathbb{C})$, that is $det(e^{tX}) = 1$.

Example: Computing the Lie Algebra of $SL(n, \mathbb{C})$

- We seek matrices such that $e^{tX} \in SL(n, \mathbb{C})$, that is $det(e^{tX}) = 1$.
- We can show that for a general X, we have that $det(e^X) = e^{tr(X)}$.

Example: Computing the Lie Algebra of $SL(n, \mathbb{C})$

- We seek matrices such that $e^{tX} \in SL(n, \mathbb{C})$, that is $det(e^{tX}) = 1$.
- We can show that for a general X, we have that $det(e^X) = e^{tr(X)}$.
- Thus, $det(e^{tX}) = e^{t \cdot tr(X)}$.

Example: Computing the Lie Algebra of $SL(n, \mathbb{C})$

- We seek matrices such that $e^{tX} \in SL(n, \mathbb{C})$, that is $det(e^{tX}) = 1$.
- We can show that for a general X, we have that $det(e^X) = e^{tr(X)}$.
- Thus, $det(e^{tX}) = e^{t \cdot tr(X)}$. So if $tr(X) = 0$ then $e^{t \cdot tr(X)} = 1$, and so $det(e^{tX}) = 1$ as well.

Example: Computing the Lie Algebra of $SL(n, \mathbb{C})$

- We seek matrices such that $e^{tX} \in SL(n, \mathbb{C})$, that is $det(e^{tX}) = 1$.
- We can show that for a general X, we have that $det(e^X) = e^{tr(X)}$.
- Thus, $det(e^{tX}) = e^{t \cdot tr(X)}$. So if $tr(X) = 0$ then $e^{t \cdot tr(X)} = 1$, and so $det(e^{tX}) = 1$ as well. So X is in the associated Lie Algebra.

Example: Computing the Lie Algebra of $SL(n, \mathbb{C})$

- We seek matrices such that $e^{tX} \in SL(n, \mathbb{C})$, that is $det(e^{tX}) = 1$.
- We can show that for a general X, we have that $det(e^{X}) = e^{tr(X)}$.
- Thus, $det(e^{tX}) = e^{t \cdot tr(X)}$. So if $tr(X) = 0$ then $e^{t \cdot tr(X)} = 1$, and so $det(e^{tX}) = 1$ as well. So X is in the associated Lie Algebra.
- Conversely, suppose that $det(e^{tX}) = 1 = e^{t \cdot tr(X)}$.

Example: Computing the Lie Algebra of $SL(n, \mathbb{C})$

- We seek matrices such that $e^{tX} \in SL(n, \mathbb{C})$, that is $det(e^{tX}) = 1$.
- We can show that for a general X, we have that $det(e^X) = e^{tr(X)}$.
- Thus, $det(e^{tX}) = e^{t \cdot tr(X)}$. So if $tr(X) = 0$ then $e^{t \cdot tr(X)} = 1$, and so $det(e^{tX}) = 1$ as well. So X is in the associated Lie Algebra.
- Conversely, suppose that $det(e^{tX}) = 1 = e^{t \cdot tr(X)}$. Then differentiating with respect to t we get that

Example: Computing the Lie Algebra of $SL(n, \mathbb{C})$

- We seek matrices such that $e^{tX} \in SL(n, \mathbb{C})$, that is $det(e^{tX}) = 1$.
- We can show that for a general X, we have that $det(e^{X}) = e^{tr(X)}$.
- Thus, $det(e^{tX}) = e^{t \cdot tr(X)}$. So if $tr(X) = 0$ then $e^{t \cdot tr(X)} = 1$, and so $det(e^{tX}) = 1$ as well. So X is in the associated Lie Algebra.
- Conversely, suppose that $det(e^{tX}) = 1 = e^{t \cdot tr(X)}$. Then differentiating with respect to t we get that

$$tr(X) = \frac{d}{dt}\left[e^{t \cdot tr(X)}\right]_{t=0} = 0$$

Example: Computing the Lie Algebra of $SL(n, \mathbb{C})$

- We seek matrices such that $e^{tX} \in SL(n, \mathbb{C})$, that is $det(e^{tX}) = 1$.
- We can show that for a general X, we have that $det(e^{X}) = e^{tr(X)}$.
- Thus, $det(e^{tX}) = e^{t \cdot tr(X)}$. So if $tr(X) = 0$ then $e^{t \cdot tr(X)} = 1$, and so $det(e^{tX}) = 1$ as well. So X is in the associated Lie Algebra.
- Conversely, suppose that $det(e^{tX}) = 1 = e^{t \cdot tr(X)}$. Then differentiating with respect to t we get that

$$tr(X) = \frac{d}{dt}\left[e^{t \cdot tr(X)}\right]_{t=0} = 0$$

- So $e^{tX} \in SL(n, \mathbb{C})$ if and only if $tr(X) = 0$.

Example: Computing the Lie Algebra of $SL(n, \mathbb{C})$

- We seek matrices such that $e^{tX} \in SL(n, \mathbb{C})$, that is $det(e^{tX}) = 1$.
- We can show that for a general X, we have that $det(e^X) = e^{tr(X)}$.
- Thus, $det(e^{tX}) = e^{t \cdot tr(X)}$. So if $tr(X) = 0$ then $e^{t \cdot tr(X)} = 1$, and so $det(e^{tX}) = 1$ as well. So X is in the associated Lie Algebra.
- Conversely, suppose that $det(e^{tX}) = 1 = e^{t \cdot tr(X)}$. Then differentiating with respect to t we get that

$$tr(X) = \frac{d}{dt}\left[e^{t \cdot tr(X)}\right]_{t=0} = 0$$

- So $e^{tX} \in SL(n, \mathbb{C})$ if and only if $tr(X) = 0$. We denote the set of traceless matrices $sl(n, \mathbb{C})$. This is the Lie Algebra of $SL(n, \mathbb{C})$!

Representations

Representations

Definition

A Representation of a Lie Algebra $\mathfrak{g}$ is a Lie Algebra Homomorphism $\pi : \mathfrak{g} \to gl(V)$. Here, a Lie Algebra Homomorphism is a linear map that preserves the bracket:

Representations

Definition

A Representation of a Lie Algebra $\mathfrak{g}$ is a Lie Algebra Homomorphism $\pi : \mathfrak{g} \to gl(V)$. Here, a Lie Algebra Homomorphism is a linear map that preserves the bracket:

$$\pi([X, Y]) = [\pi(X), \pi(Y)]$$

Representations

Definition

A Representation of a Lie Algebra $\mathfrak{g}$ is a Lie Algebra Homomorphism $\pi : \mathfrak{g} \to gl(V)$. Here, a Lie Algebra Homomorphism is a linear map that preserves the bracket:

$$\pi([X, Y]) = [\pi(X), \pi(Y)]$$

Every Lie Algebra $\mathfrak{g}$ has a natural representation given by the adjoint mapping: $\rho : \mathfrak{g} \to gl(\mathfrak{g}), \rho(X) = [X, \cdot]$.

Representations

Definition

A Representation of a Lie Algebra $\mathfrak{g}$ is a Lie Algebra Homomorphism $\pi : \mathfrak{g} \to gl(V)$. Here, a Lie Algebra Homomorphism is a linear map that preserves the bracket:

$$\pi([X, Y]) = [\pi(X), \pi(Y)]$$

Every Lie Algebra $\mathfrak{g}$ has a natural representation given by the adjoint mapping: $\rho : \mathfrak{g} \to gl(\mathfrak{g}), \rho(X) = [X, \cdot]$. Let us study this representation further.

Direct Sum Decompositions

Direct Sum Decompositions

- Let us suppose that $\mathfrak{g}$ has no nonzero abelian ideals.

Direct Sum Decompositions

- Let us suppose that $\mathfrak{g}$ has no nonzero abelian ideals.
- Then $\mathfrak{g}$ has a maximal abelian subalgebra $\mathfrak{h}$ called its Cartan subalgebra.

Direct Sum Decompositions

- Let us suppose that $\mathfrak{g}$ has no nonzero abelian ideals.
- Then $\mathfrak{g}$ has a maximal abelian subalgebra $\mathfrak{h}$ called its Cartan subalgebra.
- A nonzero $\alpha \in \mathfrak{h}$ is called a **root** if there is a nonzero $X \in \mathfrak{g}$

$$[H, X] = \overline{\alpha}(H)X = \langle \alpha, H \rangle X$$

for each $H \in \mathfrak{h}$.

Direct Sum Decompositions

- Let us suppose that $\mathfrak{g}$ has no nonzero abelian ideals.
- Then $\mathfrak{g}$ has a maximal abelian subalgebra $\mathfrak{h}$ called its Cartan subalgebra.
- A nonzero $\alpha \in \mathfrak{h}$ is called a **root** if there is a nonzero $X \in \mathfrak{g}$

$$[H, X] = \overline{\alpha}(H)X = \langle \alpha, H \rangle X$$

for each $H \in \mathfrak{h}$. The set of roots is denoted R. An X that satisfies the above is called a **root vector**.

Direct Sum Decompositions

- Let us suppose that $\mathfrak{g}$ has no nonzero abelian ideals.
- Then $\mathfrak{g}$ has a maximal abelian subalgebra $\mathfrak{h}$ called its Cartan subalgebra.
- A nonzero $\alpha \in \mathfrak{h}$ is called a **root** if there is a nonzero $X \in \mathfrak{g}$

$$[H, X] = \overline{\alpha}(H)X = \langle \alpha, H \rangle X$$

for each $H \in \mathfrak{h}$. The set of roots is denoted R. An X that satisfies the above is called a **root vector**. The set $\mathfrak{g}_\alpha$ of vectors X that satisfy the above property is called the **root space** w.r.t α.

Direct Sum Decompositions

- Let us suppose that $\mathfrak{g}$ has no nonzero abelian ideals.
- Then $\mathfrak{g}$ has a maximal abelian subalgebra $\mathfrak{h}$ called its Cartan subalgebra.
- A nonzero $\alpha \in \mathfrak{h}$ is called a **root** if there is a nonzero $X \in \mathfrak{g}$

  $$[H, X] = \overline{\alpha}(H)X = \langle \alpha, H \rangle X$$

  for each $H \in \mathfrak{h}$. The set of roots is denoted R. An X that satisfies the above is called a **root vector**. The set $\mathfrak{g}_\alpha$ of vectors X that satisfy the above property is called the **root space** w.r.t α.
- We can actually decompose $\mathfrak{g}$ into a direct sum of its root spaces

Direct Sum Decompositions

- Let us suppose that $\mathfrak{g}$ has no nonzero abelian ideals.
- Then $\mathfrak{g}$ has a maximal abelian subalgebra $\mathfrak{h}$ called its Cartan subalgebra.
- A nonzero $\alpha \in \mathfrak{h}$ is called a **root** if there is a nonzero $X \in \mathfrak{g}$

$$[H, X] = \overline{\alpha}(H)X = \langle \alpha, H \rangle X$$

  for each $H \in \mathfrak{h}$. The set of roots is denoted R. An X that satisfies the above is called a **root vector**. The set $\mathfrak{g}_\alpha$ of vectors X that satisfy the above property is called the **root space** w.r.t α.
- We can actually decompose $\mathfrak{g}$ into a direct sum of its root spaces:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

An Example in $sl(3, \mathbb{C})$

An Example in $sl(3, \mathbb{C})$

Let us choose the basis:

An Example in $sl(3, \mathbb{C})$

Let us choose the basis:

$$H_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, H_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, X_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, X_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Y_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, Y_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, Y_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

An Example in $sl(3, \mathbb{C})$

Let us choose the basis:

$$H_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, H_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, X_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, X_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Y_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, Y_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, Y_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Here, our Cartan subalgebra is $\mathfrak{h} = span\{H_1, H_2\}$.

An Example In $sl(3, \mathbb{C})$ (cont.)

An Example In $sl(3, \mathbb{C})$ (cont.)

Using the above basis, we see that

An Example In $sl(3, \mathbb{C})$ (cont.)

Using the above basis, we see that

$$[H_1, X_1] = 2X_1, [H_2, X_1] = -X_1$$

An Example In $sl(3, \mathbb{C})$ (cont.)

Using the above basis, we see that

$$[H_1, X_1] = 2X_1, [H_2, X_1] = -X_1$$

Because H_1, H_2 serve as a basis for $\mathfrak{h}$, for any $H \in \mathfrak{h}, H = aH_1 + bH_2$, and so we have

An Example In $sl(3, \mathbb{C})$ (cont.)

Using the above basis, we see that

$$[H_1, X_1] = 2X_1, [H_2, X_1] = -X_1$$

Because H_1, H_2 serve as a basis for $\mathfrak{h}$, for any $H \in \mathfrak{h}, H = aH_1 + bH_2$, and so we have

$$[H, X_1] = (2a - b)X_1$$

An Example In $sl(3, \mathbb{C})$ (cont.)

Using the above basis, we see that

$$[H_1, X_1] = 2X_1, [H_2, X_1] = -X_1$$

Because H_1, H_2 serve as a basis for $\mathfrak{h}$, for any $H \in \mathfrak{h}, H = aH_1 + bH_2$, and so we have

$$[H, X_1] = (2a - b)X_1$$

So X_1 is a root vector, corresponding to the root $\overline{\alpha}(H) = \overline{\alpha}(aH_1 + bH_2) = 2a - b$.

Root Systems

Root Systems

Definition

A root system (E, R) is a finite-dimensional real vector space E with an inner product $\langle \cdot, \cdot \rangle$ together with a finite set of nonzero vectors $R \subseteq E$ such that

Root Systems

Definition

A root system (E, R) is a finite-dimensional real vector space E with an inner product $\langle \cdot, \cdot \rangle$ together with a finite set of nonzero vectors $R \subseteq E$ such that

- R spans E

Root Systems

Definition

A root system (E, R) is a finite-dimensional real vector space E with an inner product $\langle \cdot, \cdot \rangle$ together with a finite set of nonzero vectors $R \subseteq E$ such that

- R spans E
- If $\alpha \in R$ and $c \in \mathbb{R}$, then $c\alpha \in R$ only if $c = \pm 1$

Root Systems

Definition

A root system (E, R) is a finite-dimensional real vector space E with an inner product $\langle \cdot, \cdot \rangle$ together with a finite set of nonzero vectors $R \subseteq E$ such that

- R spans E
- If $\alpha \in R$ and $c \in \mathbb{R}$, then $c\alpha \in R$ only if $c = \pm 1$
- If $\alpha, \beta \in R$, then so is

Root Systems

Definition

A root system (E, R) is a finite-dimensional real vector space E with an inner product $\langle \cdot, \cdot \rangle$ together with a finite set of nonzero vectors $R \subseteq E$ such that

- R spans E
- If $\alpha \in R$ and $c \in \mathbb{R}$, then $c\alpha \in R$ only if $c = \pm 1$
- If $\alpha, \beta \in R$, then so is

$$\beta - 2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha$$

Root Systems

Definition

A root system (E, R) is a finite-dimensional real vector space E with an inner product $\langle \cdot, \cdot \rangle$ together with a finite set of nonzero vectors $R \subseteq E$ such that

- R spans E
- If $\alpha \in R$ and $c \in \mathbb{R}$, then $c\alpha \in R$ only if $c = \pm 1$
- If $\alpha, \beta \in R$, then so is

$$\beta - 2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha$$

- For every $\alpha, \beta \in R$,

Root Systems

Definition

A root system (E, R) is a finite-dimensional real vector space E with an inner product $\langle \cdot, \cdot \rangle$ together with a finite set of nonzero vectors $R \subseteq E$ such that

- R spans E
- If $\alpha \in R$ and $c \in \mathbb{R}$, then $c\alpha \in R$ only if $c = \pm 1$
- If $\alpha, \beta \in R$, then so is

$$\beta - 2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha$$

- For every $\alpha, \beta \in R$,

$$2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$$

Possible Root Configurations

Possible Root Configurations

Suppose α, β are roots that are not colinear and θ is the angle between them. Further, suppose $\langle \alpha, \alpha \rangle \geq \langle \beta, \beta \rangle$.

Possible Root Configurations

Suppose α, β are roots that are not colinear and θ is the angle between them. Further, suppose $\langle\alpha, \alpha\rangle \geq \langle\beta, \beta\rangle$. Then one of the following is true:

Possible Root Configurations

Suppose α, β are roots that are not colinear and θ is the angle between them. Further, suppose $\langle \alpha, \alpha \rangle \geq \langle \beta, \beta \rangle$. Then one of the following is true:

- $\langle \alpha, \beta \rangle = 0$.

Possible Root Configurations

Suppose α, β are roots that are not colinear and θ is the angle between them. Further, suppose $\langle \alpha, \alpha \rangle \geq \langle \beta, \beta \rangle$. Then one of the following is true:

- $\langle \alpha, \beta \rangle = 0$.
- $\langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle$ and $\theta = \frac{\pi}{3}, \frac{2\pi}{3}$.

Possible Root Configurations

Suppose α, β are roots that are not colinear and θ is the angle between them. Further, suppose $\langle \alpha, \alpha \rangle \geq \langle \beta, \beta \rangle$. Then one of the following is true:

- $\langle \alpha, \beta \rangle = 0$.
- $\langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle$ and $\theta = \frac{\pi}{3}, \frac{2\pi}{3}$.
- $\langle \alpha, \alpha \rangle = 2\langle \beta, \beta \rangle$ and $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$.

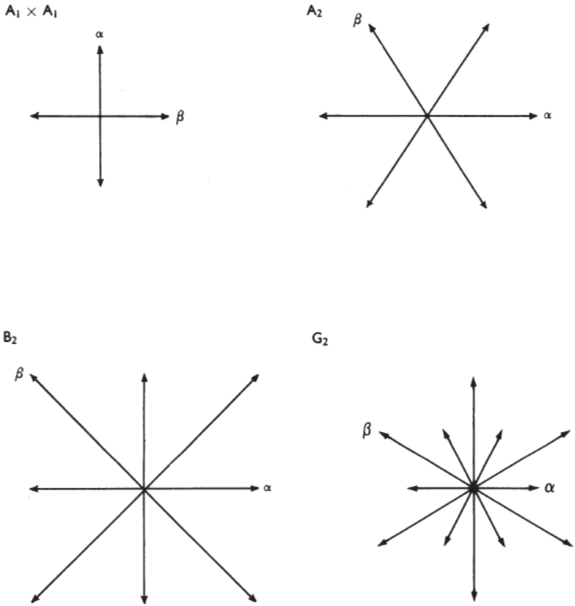
Possible Root Configurations

Suppose α, β are roots that are not colinear and θ is the angle between them. Further, suppose $\langle \alpha, \alpha \rangle \geq \langle \beta, \beta \rangle$. Then one of the following is true:

- $\langle \alpha, \beta \rangle = 0$.
- $\langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle$ and $\theta = \frac{\pi}{3}, \frac{2\pi}{3}$.
- $\langle \alpha, \alpha \rangle = 2\langle \beta, \beta \rangle$ and $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$.
- $\langle \alpha, \alpha \rangle = 3\langle \beta, \beta \rangle$ and $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$.

Rank 2 Root Configurations

Rank 2 Root Configurations



General Root Configurations

General Root Configurations

Definition

Given a root system $(E, R), \Delta \subseteq R$ is a **base** if

General Root Configurations

Definition

Given a root system $(E, R), \Delta \subseteq R$ is a **base** if

- Δ is a basis for E

General Root Configurations

Definition

Given a root system $(E, R), \Delta \subseteq R$ is a **base** if

- Δ is a basis for E
- Each $\alpha \in R$ can be expressed as a linear combination of elements of Δ with either all non-negative or non-positive integer coefficients.

General Root Configurations

Definition

Given a root system $(E, R), \Delta \subseteq R$ is a **base** if

- Δ is a basis for E
- Each $\alpha \in R$ can be expressed as a linear combination of elements of Δ with either all non-negative or non-positive integer coefficients.

Elements of Δ are called positive simple roots.

General Root Configurations

Definition

Given a root system $(E, R), \Delta \subseteq R$ is a **base** if

- Δ is a basis for E
- Each $\alpha \in R$ can be expressed as a linear combination of elements of Δ with either all non-negative or non-positive integer coefficients.

Elements of Δ are called positive simple roots.

Given a base $\Delta = \{\alpha_1, \dots, \alpha_n\}$, we can understand the root system via the values of $\langle \alpha_i, \alpha_j \rangle$ for $i \neq j$ and the relative sizes of the $||\alpha_i||$.

General Root Configurations

Definition

Given a root system $(E, R), \Delta \subseteq R$ is a **base** if

- Δ is a basis for E
- Each $\alpha \in R$ can be expressed as a linear combination of elements of Δ with either all non-negative or non-positive integer coefficients.

Elements of Δ are called positive simple roots.

Given a base $\Delta = \{\alpha_1, \ldots, \alpha_n\}$, we can understand the root system via the values of $\langle \alpha_i, \alpha_j \rangle$ for $i \neq j$ and the relative sizes of the $||\alpha_i||$. These are encoded in Dynkin Diagrams.

Dynkin Diagrams

Dynkin Diagrams

- Set a vertex i for each $\alpha_i \in \Delta$.

Dynkin Diagrams

- Set a vertex i for each $\alpha_i \in \Delta$.
- The number of edges connected vertex i and vertex j is equal to the value of $\langle \alpha_i, \alpha_j \rangle$.

Dynkin Diagrams

- Set a vertex i for each $\alpha_i \in \Delta$.
- The number of edges connected vertex i and vertex j is equal to the value of $\langle \alpha_i, \alpha_j \rangle$. Namely, this encodes the information about the angle between α_i and α_j.

Dynkin Diagrams

- Set a vertex i for each $\alpha_i \in \Delta$.
- The number of edges connected vertex i and vertex j is equal to the value of $\langle \alpha_i, \alpha_j \rangle$. Namely, this encodes the information about the angle between α_i and α_j. For instance, if there is one edge between α_1 and α_2 then the angle between them is $\frac{2\pi}{3}$.

Dynkin Diagrams

- Set a vertex i for each $\alpha_i \in \Delta$.
- The number of edges connected vertex i and vertex j is equal to the value of $\langle \alpha_i, \alpha_j \rangle$. Namely, this encodes the information about the angle between α_i and α_j. For instance, if there is one edge between α_1 and α_2 then the angle between them is $\frac{2\pi}{3}$.
- Add arrows ($>$ or $<$) on the edges connecting vertices i and j to encode whether $||\alpha_i|| > ||\alpha_j||$.

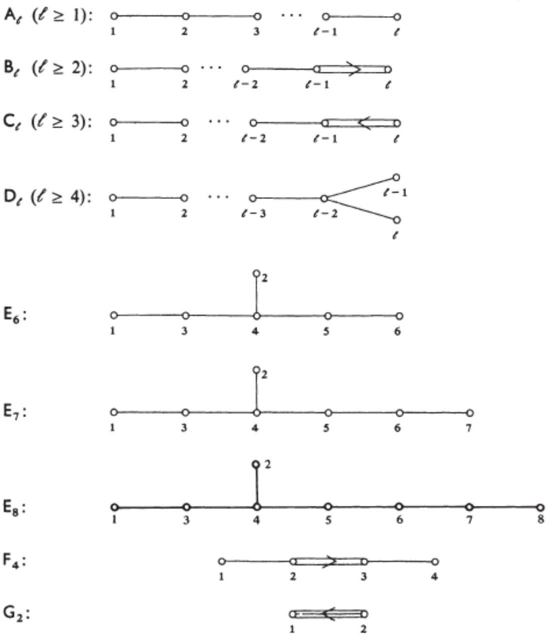
Dynkin Diagrams

- Set a vertex i for each $\alpha_i \in \Delta$.
- The number of edges connected vertex i and vertex j is equal to the value of $\langle \alpha_i, \alpha_j \rangle$. Namely, this encodes the information about the angle between α_i and α_j. For instance, if there is one edge between α_1 and α_2 then the angle between them is $\frac{2\pi}{3}$.
- Add arrows ($>$ or $<$) on the edges connecting vertices i and j to encode whether $||\alpha_i|| > ||\alpha_j||$.

Theorem

If (E, R) is a root system with $dim(E) = \ell$, then its Dynkin Diagram is one of the following.

Dynkin Diagrams (cont.)



Lie Algebras and Fantastic Results

Lie Algebras and Fantastic Results

- A Lie Algebra's roots correspond to a root system.

Lie Algebras and Fantastic Results

- A Lie Algebra's roots correspond to a root system. Namely, $(E, R) = (\mathfrak{h}, R)$.

Lie Algebras and Fantastic Results

- A Lie Algebra's roots correspond to a root system. Namely, $(E, R) = (\mathfrak{h}, R)$.
- We can classify root systems

Lie Algebras and Fantastic Results

- A Lie Algebra's roots correspond to a root system. Namely, $(E, R) = (\mathfrak{h}, R)$.
- We can classify root systems, so we can classify semisimple Lie Algebras

Lie Algebras and Fantastic Results

- A Lie Algebra's roots correspond to a root system. Namely, $(E, R) = (\mathfrak{h}, R)$.
- We can classify root systems, so we can classify semisimple Lie Algebras
- "Fantastic Theorem"

Lie Algebras and Fantastic Results

- A Lie Algebra's roots correspond to a root system. Namely, $(E, R) = (\mathfrak{h}, R)$.
- We can classify root systems, so we can classify semisimple Lie Algebras
- "Fantastic Theorem": given an abstract root system, there is a unique (up to isomorphism) semisimple Lie Algebra over $\mathbb{C}$ which has this abstract system as its root system.

Lie Algebras and Fantastic Results

- A Lie Algebra's roots correspond to a root system. Namely, $(E, R) = (\mathfrak{h}, R)$.
- We can classify root systems, so we can classify semisimple Lie Algebras
- "Fantastic Theorem": given an abstract root system, there is a unique (up to isomorphism) semisimple Lie Algebra over $\mathbb{C}$ which has this abstract system as its root system.
- Two semisimple, complex Lie Algebras are isomorphic if and only if their root systems are isomorphic!

General Representations

General Representations

- Throughout this talk we studied the "natural" representation $\rho(X) = [X, \cdot]$.

General Representations

- Throughout this talk we studied the "natural" representation $\rho(X) = [X, \cdot]$.
- The above discussion generalizes to any representation $\pi : \mathfrak{g} \to gl(V)$.

General Representations

- Throughout this talk we studied the "natural" representation $\rho(X) = [X, \cdot]$.
- The above discussion generalizes to any representation $\pi : \mathfrak{g} \to gl(V)$.
- Roots become weights, we can decompose V into a direct sum of weight spaces

General Representations

- Throughout this talk we studied the "natural" representation $\rho(X) = [X, \cdot]$.
- The above discussion generalizes to any representation $\pi : \mathfrak{g} \to gl(V)$.
- Roots become weights, we can decompose V into a direct sum of weight spaces

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$$

General Representations

- Throughout this talk we studied the "natural" representation $\rho(X) = [X, \cdot]$.
- The above discussion generalizes to any representation $\pi : \mathfrak{g} \to gl(V)$.
- Roots become weights, we can decompose V into a direct sum of weight spaces

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$$

weights will have certain configurations, etc.

General Representations

- Throughout this talk we studied the "natural" representation $\rho(X) = [X, \cdot]$.
- The above discussion generalizes to any representation $\pi : \mathfrak{g} \to gl(V)$.
- Roots become weights, we can decompose V into a direct sum of weight spaces

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$$

  weights will have certain configurations, etc.
- This allows us to classify the representations of a Lie Algebra $\mathfrak{g}$ to a finite-dimensional vector space V.

General Representations

- Throughout this talk we studied the "natural" representation $\rho(X) = [X, \cdot]$.
- The above discussion generalizes to any representation $\pi : \mathfrak{g} \to gl(V)$.
- Roots become weights, we can decompose V into a direct sum of weight spaces

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$$

  weights will have certain configurations, etc.
- This allows us to classify the representations of a Lie Algebra $\mathfrak{g}$ to a finite-dimensional vector space V.
- Through study of the weight lattice and root lattice (the $\mathbb{Z}$-span of the weights/roots respectively)

General Representations

- Throughout this talk we studied the "natural" representation $\rho(X) = [X, \cdot]$.
- The above discussion generalizes to any representation $\pi : \mathfrak{g} \to gl(V)$.
- Roots become weights, we can decompose V into a direct sum of weight spaces

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$$

  weights will have certain configurations, etc.
- This allows us to classify the representations of a Lie Algebra $\mathfrak{g}$ to a finite-dimensional vector space V.
- Through study of the weight lattice and root lattice (the $\mathbb{Z}$-span of the weights/roots respectively), we have that

$$\text{weight lattice/root lattice}$$

  is a finite group

General Representations

- Throughout this talk we studied the "natural" representation $\rho(X) = [X, \cdot]$.
- The above discussion generalizes to any representation $\pi : \mathfrak{g} \to gl(V)$.
- Roots become weights, we can decompose V into a direct sum of weight spaces

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$$

weights will have certain configurations, etc.

- This allows us to classify the representations of a Lie Algebra $\mathfrak{g}$ to a finite-dimensional vector space V.
- Through study of the weight lattice and root lattice (the $\mathbb{Z}$-span of the weights/roots respectively), we have that

weight lattice/root lattice

is a finite group

References

- Brian C. Hall. Lie Groups, Lie Algebras, and Representations, 2 ed. Springer, 2015.
- James E. Humphreys. Introduction to Lie Algebras and Representation Theory. Springer, 1972.
- Jeffrey Adams. Computer Computations in Representation Theory II: Root Systems and Weyl Groups.