Classification of Semisimple Lie Algebras

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What is a Lie Group?

Definition

A Lie Group $G$ is a group that is also a differentiable manifold such that its group operations are smooth.

Example

The set of matrices $\text{GL}(n, \mathbb{C})$ is a Lie Group under matrix multiplication. So is $\text{SL}(n, \mathbb{C})$, and $\text{U}(n)$, $\text{SO}(n)$, and more.

Definition

A Matrix Lie Group is a closed subgroup $G \subseteq \text{GL}(n, \mathbb{C})$. That is, whenever $\{A_n\} \subseteq G$ converges to $A$, then either $A \in G$ or $A \notin \text{GL}(n, \mathbb{C})$.

For example: $\text{SL}(n, \mathbb{C})$ is a Matrix Lie Group because it is a subgroup of $\text{GL}(n, \mathbb{C})$, and if $\{A_n\} \subseteq \text{SL}(n, \mathbb{C})$ converges to $A$, then $A \in \text{SL}(n, \mathbb{C})$ because each $A_n$ has determinant one and the determinant function is continuous.
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Tangent Spaces and Lie Algebras

Definition

A Lie Algebra $g$ is a vector space along with a map $[·, ·] : g \times g \rightarrow g$ that is bilinear, skew symmetric, and satisfies the Jacobi Identity:

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

Two elements $X, Y \in g$ are said to commute if $[X, Y] = 0$.

▸ Given a differentiable manifold $M$ and a point $p \in M$, the set of tangent vectors at $p$ is denoted $T_p(M)$.

▸ Every Lie Group $G$ has an associated Lie Algebra $g = T_0G$, where $0 \in G$ is the identity element.

Example

$\mathbb{R}^3$ with $[x, y] = x \times y$ is also a Lie Algebra.

$\mathfrak{gl}(V)$, the set of linear maps from $V$ to itself, is a Lie Algebra with bracket $[x, y] = xy - yx$. 
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$\mathbb{R}^3$ with $[x, y] = x \times y$ is also a Lie Algebra. $\text{gl}(V)$, the set of linear maps from $V$ to itself, is a Lie Algebra with bracket $[x, y] = xy - yx$. 
Recall the matrix exponential map:

\[ e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} \]

It can be shown that this mapping converges for any complex-valued matrix \( A \), and is in fact continuous. In the case that \( G \) is a Matrix Lie Group, the Lie Algebra of \( G \) can be computed more practically as the set of complex matrices \( X \) such that \( e^{tX} \in G \) for every real \( t \).
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Example: Computing the Lie Algebra of $SL(n, \mathbb{C})$ 

We seek matrices such that $e^{tX} \in SL(n, \mathbb{C})$, that is $\det(e^{tX}) = 1$.

We can show that for a general $X$, we have that $\det(e^{X}) = e^{\text{tr}(X)}$.

Thus, $\det(e^{tX}) = e^{t \cdot \text{tr}(X)}$.

So if $\text{tr}(X) = 0$ then $e^{t \cdot \text{tr}(X)} = 1$, and so $\det(e^{tX}) = 1$ as well. So $X$ is in the associated Lie Algebra.

Conversely, suppose that $\det(e^{tX}) = 1 = e^{t \cdot \text{tr}(X)}$.

Then differentiating with respect to $t$ we get that $\text{tr}(X) = \frac{d}{dt}[e^{t \cdot \text{tr}(X)}]_{t=0} = 0$.

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We denote the set of traceless matrices $\text{sl}(n, \mathbb{C})$. This is the Lie Algebra of $SL(n, \mathbb{C})$!
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$\det$ represents the determinant.
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Representations

A Representation of a Lie Algebra $g$ is a Lie Algebra Homomorphism $\pi: g \rightarrow gl(V)$. Here, a Lie Algebra Homomorphism is a linear map that preserves the bracket:

$$\pi([X, Y]) = [\pi(X), \pi(Y)].$$

Every Lie Algebra $g$ has a natural representation given by the adjoint mapping:

$$\rho: g \rightarrow gl(g), \rho(X) = [X, \cdot].$$

Let us study this representation further.
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Direct Sum Decompositions

Let us suppose that $g$ has no nonzero abelian ideals. Then $g$ has a maximal abelian subalgebra $h$ called its Cartan subalgebra. A nonzero $\alpha \in h$ is called a root if there is a nonzero $X \in g$ such that $[H,X] = \alpha(H)X = \langle \alpha, H \rangle X$ for each $H \in h$. The set of roots is denoted $R$. An $X$ that satisfies the above is called a root vector. The set $g_\alpha$ of vectors $X$ that satisfy the above property is called the root space w.r.t $\alpha$.

We can actually decompose $g$ into a direct sum of its root spaces:

$$g = h \oplus \bigoplus_{\alpha \in R} g_\alpha$$
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An Example in $sl(3, \mathbb{C})$
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Let us choose the basis:

$$H_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$H_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

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An Example in $sl(3, \mathbb{C})$

Let us choose the basis:

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An Example In $sl(3, \mathbb{C})$ (cont.)

Using the above basis, we see that 
\[
\begin{align*}
[H_1, X_1] &= 2X_1, \\
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\end{align*}
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Because $H_1, H_2$ serve as a basis for $h$, for any $H \in h$, $H = aH_1 + bH_2$, and so we have 
\[
[H, X_1] = (2a - b)X_1
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So $X_1$ is a root vector, corresponding to the root $\alpha(H) = \alpha(aH_1 + bH_2) = 2a - b$. 
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Root Systems

Definition

A root system \((E, R)\) is a finite-dimensional real vector space \(E\) with an inner product \(\langle \cdot, \cdot \rangle\) together with a finite set of nonzero vectors \(R \subseteq E\) such that

- \(R\) spans \(E\)
- If \(\alpha \in R\) and \(c \in R\), then \(c\alpha \in R\) only if \(c = \pm 1\)
- If \(\alpha, \beta \in R\), then so is \(\beta - 2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha\)
- For every \(\alpha, \beta \in R\), \(2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}\)
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Possible Root Configurations

Suppose \( \alpha, \beta \) are roots that are not colinear and \( \theta \) is the angle between them. Further, suppose \( \langle \alpha, \alpha \rangle \geq \langle \beta, \beta \rangle \).

Then one of the following is true:

\[ \langle \alpha, \beta \rangle = 0. \]

\[ \langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle \text{ and } \theta = \frac{\pi}{3}, \frac{2\pi}{3}. \]

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Rank 2 Root Configurations

\[ A_1 \times A_1 \]
\[ A_2 \]
\[ B_2 \]
\[ G_2 \]
General Root Configurations

Definition

Given a root system \((E, R)\), \(\Delta \subseteq R\) is a base if
\(\nabla \Delta\) is a basis for \(E\). Each \(\alpha \in R\) can be expressed as a linear combination of elements of \(\Delta\) with either all non-negative or non-positive integer coefficients.

Elements of \(\Delta\) are called positive simple roots.

Given a base \(\Delta = \{\alpha_1, \ldots, \alpha_n\}\), we can understand the root system via the values of \(\langle \alpha_i, \alpha_j \rangle\) for \(i \neq j\) and the relative sizes of \(||\alpha_i||\).

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Dynkin Diagrams

- Set a vertex $i$ for each $\alpha_i \in \Delta$.
- The number of edges connected vertex $i$ and vertex $j$ is equal to the value of $\langle \alpha_i, \alpha_j \rangle$.
  - Namely, this encodes the information about the angle between $\alpha_i$ and $\alpha_j$.
  - For instance, if there is one edge between $\alpha_1$ and $\alpha_2$ then the angle between them is $\frac{2\pi}{3}$.
- Add arrows ($>$ or $<$) on the edges connecting vertices $i$ and $j$ to encode whether $||\alpha_i|| > ||\alpha_j||$.

Theorem

If $(E, R)$ is a root system with $\dim(E) = \ell$, then its Dynkin Diagram is one of the following.
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Dynkin Diagrams (cont.)

$A_\ell$ ($\ell \geq 1$):

$B_\ell$ ($\ell \geq 2$):

$C_\ell$ ($\ell \geq 3$):

$D_\ell$ ($\ell \geq 4$):

$E_6$:

$E_7$:

$E_8$:

$F_4$:

$G_2$: 
Lie Algebras and Fantastic Results

A Lie Algebra's roots correspond to a root system. Namely, 

\((E, R) = (h, R)\).

We can classify root systems, so we can classify semisimple Lie Algebras.

"Fantastic Theorem": given an abstract root system, there is a unique (up to isomorphism) semisimple Lie Algebra over \(\mathbb{C}\) which has this abstract system as its root system.

Two semisimple, complex Lie Algebras are isomorphic if and only if their root systems are isomorphic!
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General Representations

Throughout this talk we studied the "natural" representation $\rho(X) = [X, \cdot]$. The above discussion generalizes to any representation $\pi: g \rightarrow \mathfrak{gl}(V)$.

Roots become weights, we can decompose $V$ into a direct sum of weight spaces $V = \bigoplus_{\lambda \in h^*} V_\lambda$. Weights will have certain configurations, etc.

This allows us to classify the representations of a Lie Algebra $g$ to a finite-dimensional vector space $V$. Through study of the weight lattice and root lattice (the $\mathbb{Z}$-span of the weights/roots respectively), we have that weight lattice/root lattice is a finite group.
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Roots become weights, we can decompose \( V \) into a direct sum of weight spaces
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weights will have certain configurations, etc.

This allows us to classify the representations of a Lie Algebra \( \mathfrak{g} \) to a finite-dimensional vector space \( V \).
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References