The Glue Function
An Example of How Mathematicians Think

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1. Function and sequence: different properties

2. Introducing... the glue function!

3. Let’s make our function smooth
Problem

*Find a continuous function* \( f : \mathbb{R} \rightarrow \mathbb{R} \) *such that* 
\[ \lim_{n \to +\infty} f(n) = \infty, \text{ but } \lim_{x \to +\infty} f(x) \text{ does not exist.} \]
Let's make our function smooth!

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One function that would have such properties would be one that...

- takes consistently increasing values — let’s say, for instance, \( f(n) = n \) — at all natural numbers
- takes a constant value — let’s say, 1 — at all half values
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Introducing... the glue function!

Function and sequence: different properties

Let’s make our function smooth

One way to do this is to attach all the points with straight lines.

\[ \forall n \in \mathbb{N}, f(x) = \begin{cases} 1 & \text{if } x = n - \frac{1}{2} \ \frac{2n - 2}{x^2} + 3n & \text{if } n - \frac{1}{2} < x < n \ \frac{1}{2} + \frac{2n - n}{x} & \text{if } x = n + \frac{1}{2} \end{cases} \]
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**Solution**

\[
\forall n \in \mathbb{N}, f(x) = \begin{cases} 
1 & \text{if } x = n - \frac{1}{2} \\
(2n - 2)x - 2n^2 + 3n & \text{if } n - \frac{1}{2} < x < n \\
n & \text{if } x = n \\
(2 - 2n)x + 2n^2 - n & \text{if } n < x < n - \frac{1}{2} \\
1 & \text{if } x = n + \frac{1}{2}
\end{cases}
\]
Introducing... the glue function!

Let's make our function smooth.

Figure: A zigzag function.
This function is continuous, but it is not smooth.
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**Definition (smooth)**

A function is **smooth** if it is infinitely differentiable.
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**Definition (glue function)**

A **glue function** or interpolating function is a function that glues together piecewise functions smoothly.
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**Definition (glue function)**

A **glue function** or interpolating function is a function that glues together piecewise functions smoothly.

**Theorem**

*There exists a smooth glue function* $\chi_1$ *such that*

$$\chi_1(x) = \begin{cases} 
0 & \text{if } x < -1 \\
1 & \text{if } x > 1 
\end{cases}$$
Introducing... the glue function!

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**Figure:** The glue function $\chi_1(x)$. 

![Graph of the glue function $\chi_1(x)$]
Problem

*Find a smooth function* $f : \mathbb{R} \to \mathbb{R}$ *such that*

$$f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ \sin x & \text{if } x > 2 \end{cases}$$
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Use a one-half scale of the glue function to make this piecewise function smooth.
Problem

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Solution

$$f(x) = x^2 \cdot \left(1 - \chi_{\frac{1}{2}}(x - 1.5)\right) + \sin x \cdot \chi_{\frac{1}{2}}(x - 1.5)$$
Introducing the glue function!

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Figure: A smooth zigzag function.