The Glue Function

An Example of How Mathematicians Think

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Problem

Find a continuous function $f : \mathbb{R} \to \mathbb{R}$ *such that* $\lim_{n \to +\infty} f(n) = \infty$, *but* $\lim_{x \to +\infty} f(x)$ *does not exist.*

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One function that would have such properties would be one that...

- takes consistently increasing values — let's say, for instance, $f(n) = n$ — at all natural numbers
- takes a constant value — let's say, 1 — at all half values

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Solution

$$\forall n \in \mathbb{N}, f(x) = \begin{cases} 1 & \textit{if } x = n - \frac{1}{2} \\ (2n-2)x - 2n^2 + 3n & \textit{if } n - \frac{1}{2} < x < n \\ n & \textit{if } x = n \\ (2-2n)x + 2n^2 - n & \textit{if } n < x < n - \frac{1}{2} \\ 1 & \textit{if } x = n + \frac{1}{2} \end{cases}$$

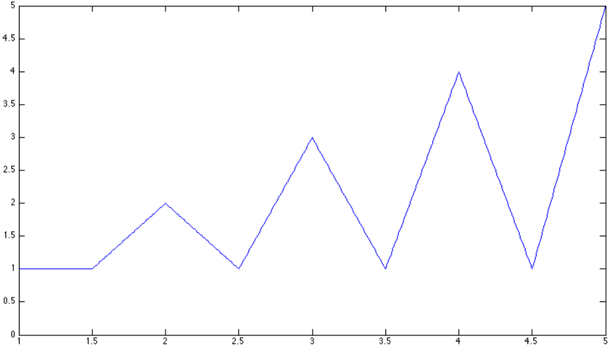

Figure: A zigzag function.

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Theorem

There exists a smooth glue function χ_1 such that

$$\chi_1(x) = \begin{cases} 0 & \text{if } x < -1 \\ 1 & \text{if } x > 1 \end{cases}$$

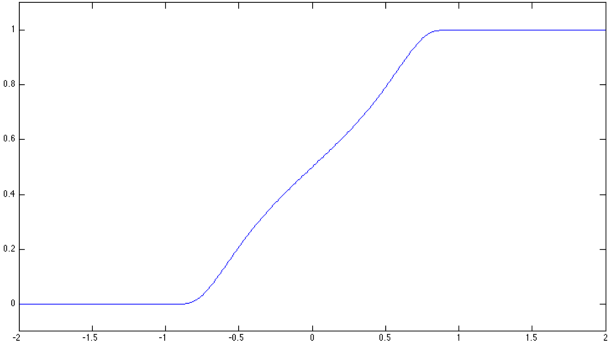

Figure: The glue function $\chi_1(x)$.

Problem

Find a smooth function $f : \mathbb{R} \to \mathbb{R}$ *such that*

$$f(x) = \begin{cases} x^2 & \textit{if } x < 1 \\ \sin x & \textit{if } x > 2 \end{cases}$$

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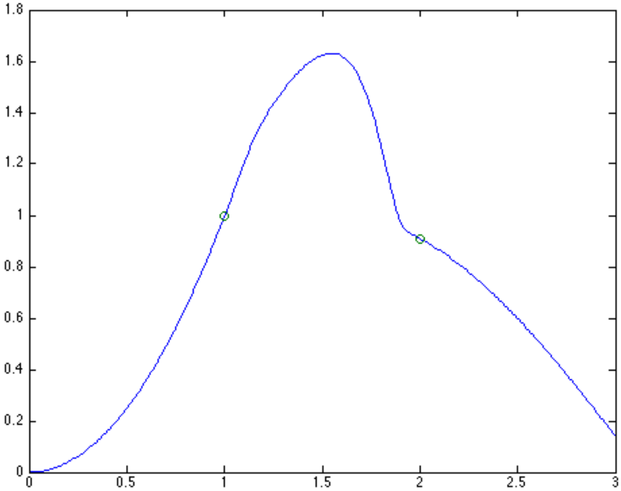
Solution

$$f(x) = x^2 \cdot (1 - \chi_{\frac{1}{2}}(x - 1.5)) + \sin x \cdot \chi_{\frac{1}{2}}(x - 1.5)$$


1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
2.5
3

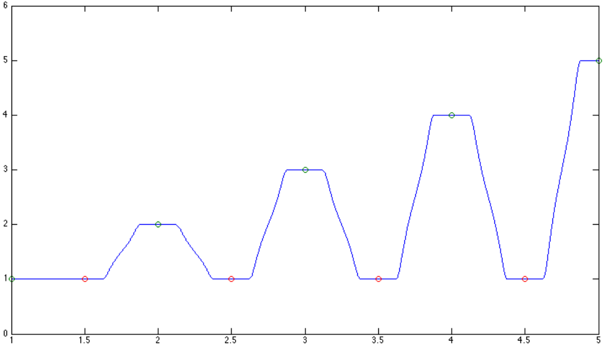


Figure: A smooth zigzag function.