Elementary Topological Properties of R^p

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Bolzano-Weierstrass Theorem:

Every bounded infinite subset of R^p has a cluster point.

Terminology

Cluster point:

A limit point (or cluster point or accumulation point) of a set S in a topological space X is a point x that can be "approximated" by points of S in the sense that every neighbourhood of x with respect to the topology on X also contains a point of S other than x itself.

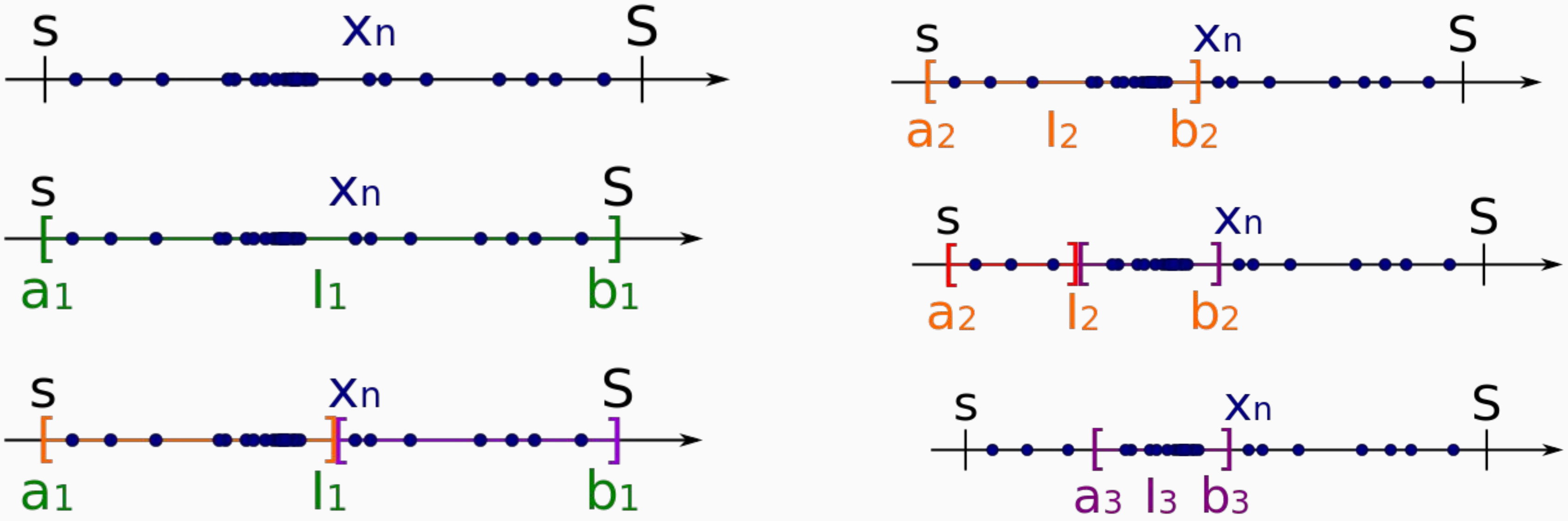
Cell:

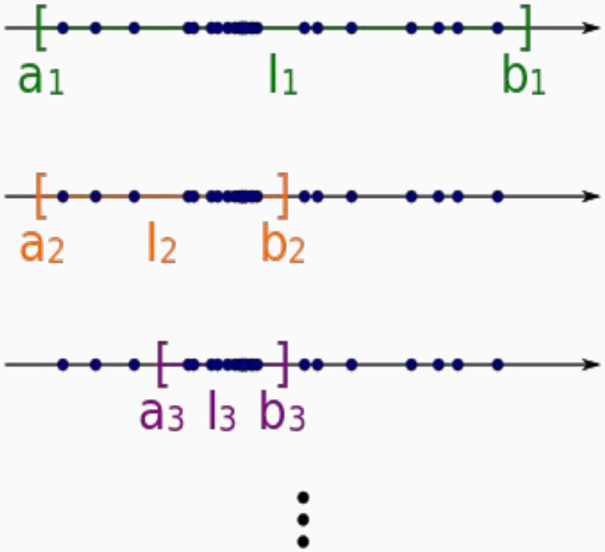
It is the Cartesian product of k closed intervals on the real line

Nested Cells Theorem

10.2 NESTED CELLS THEOREM. *Let* (I_k) *be a sequence of non-empty closed cells in* $\mathbf{R}^p$ *which is nested in the sense that* $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_k \supseteq \cdots$. *Then there exists a point in* $\mathbf{R}^p$ *which belongs to all of the cells.*

Outline of Bolzano-Weierstrass




a1
I1
b1
a2
I2
b2
a3
I3
b3

Proof of Bolzano-Weierstrass

- Begin with a bounded set with infinite number of elements
- Have a closed cell I1 containing B
- Bisect I1 into 2^p closed cells
- Find subcell with infinitely many points, calling it I2 and bisecting it
- Develop a nested sequence I_k of non-empty closed cells in R^p
- Apply Nested Cells Theorem
- Show y is a cluster point

Heine-Borel Theorem

For a subset S of Euclidean space R^n, the following two statements are equivalent:

- S is closed and bounded
- S is compact, that is, every open cover of S has a finite subcover.

Proof of Heine-Borel Theorem

- Given a set K, K closed and bounded => K compact
- Prove by contradiction:
  - Suppose G is an infinite union of sets covering K. One of the 2^p subcells contains a point not covered by any finite subcover of G
  - Apply Bolzano-Weierstrass