Using Residue Theory to Evaluate Infinite Sums

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Definitions

- A function $f : D \to \mathbb{C}$ is called **Holomorphic** if it is complex-differentiable in the domain D.
- A function $f : D \to \mathbb{C}$ is called **Analytic** if it has a convergent power series about some point $z_0 \in D$.
- A function has a **Singularity** at a point if it fails to be well defined at that point, or appears to “blow up” at that point.

We also use some well-known results from complex analysis.

$$e^{i\pi} + 1 = 0$$

Laurent Series: Generalizing Power Series in $\mathbb{C}$

- We are used to using Taylor series to describe a differentiable function as a convergent series.
- Suppose that we have a function analytic in some neighborhood around a point **but not necessarily at that point**.
- Then we write

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

Complex Integration

Consider the problem of finding $\int_{C_r}(z-z_0)^n dz$ where $\mathbf{C}$ is the closed contour around a circle given by $C_r : z(t) = z_0 + re^{it}$ for $t \in [0, 2\pi]$.

$$\int_{C_r}(z-z_0)^n dz = \int_0^{2\pi} r^n e^{int} ire^{it} dt = ir^{n+1} \int_0^{2\pi} e^{it(n+1)} dt$$

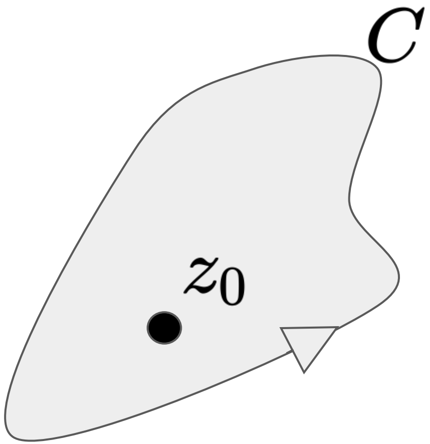
- For $n \neq -1$:

$$ir^{n+1} \int_0^{2\pi} e^{it(n+1)} dt = \frac{r^{n+1} e^{it(n+1)}}{(n+1)}\Big|_0^{2\pi} = r^{n+1}\left(\frac{1}{n+1} - \frac{1}{n+1}\right) = 0$$

- For $n = -1$:

$$ir^{n+1} \int_0^{2\pi} e^{it(n+1)} dt = it\Big|_0^{2\pi} = 2\pi i$$

Residue Theory



We want an easier way to perform complex integration. Recall

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

Integrating both sides, we find that almost every term vanishes.

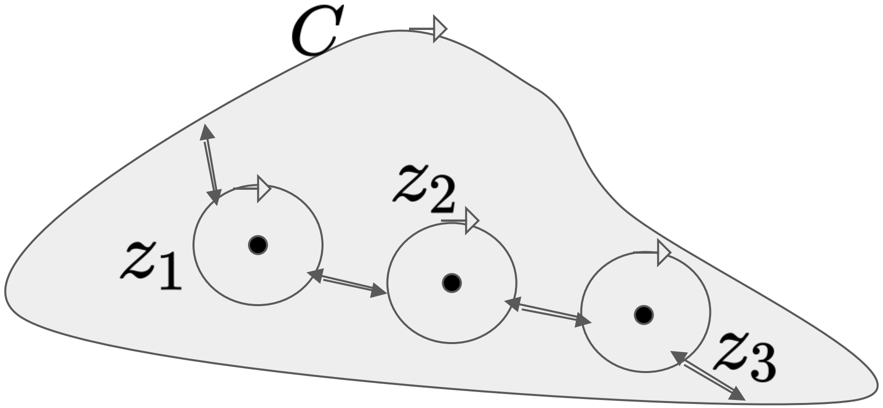
$$\int_C f(z)dz = \int_C a_0 + a_1(z-z_0)^1 + a_{-1}(z-z_0)^{-1} + a_2(z-z_0)^2 + a_{-2}(z-z_0)^{-2} + \ldots dz$$
$$= 0 + 0 + a_{-1}(2\pi i) + 0 + 0 + \ldots = 2\pi i a_{-1}$$

We therefore call the negative first coefficient in a function's Laurent Series its **Residue** at z_0.

Cauchy’s Residue Theorem

For a function having residues $z_0, z_1, \ldots, z_k$ inside a contour, we have the following formula.

$$\int_c f(z)dz = 2\pi i \sum_{j=0}^{k} Res(f, z_k)$$



Calculating Residues

We want to find a formula to calculate the residues of a function with the form:

$$f(z) = a_{-m}(z-z_0)^{-m} + a_{-m+1}(z-z_0)^{-m+1} + \ldots + a_{-1}(z-z_0)^{-1} + a_0 + a_1(z-z_0) + \ldots$$

Multiplying by $(z-z_0)^m$, taking $m-1$ derivatives, and the limit approaching z_0 :

$$((z-z_0)^m f(z))^{(m-1)} = a_{-1}(m-1)! + (m-1)!a_0(z-z_0) + a_1(m-1)!(z-z_0)^2 + \ldots$$

This gives us a formula in terms of $m, z_0,$ and derivatives of $f(z)$

$$\frac{1}{(m-1)!} \lim_{z \to z_0} ((z-z_0)^m f(z))^{(m-1)} = Res(f(z), z_0)$$

Using the Residue Theorem: Part 1

Let $g(z) = \pi f(z) \cot(\pi z)$, where f is a function decaying like $1/z^2$.

We note that this function has singularities at all the integers, so we use our formula to calculate the residues at these points.

$$
\begin{aligned}
Res(g,k) &= \lim_{z \to k} (z-k)\pi f(z) \frac{\cos(\pi z)}{\sin(\pi z)} \\
&= \lim_{z \to k} \frac{\pi(z-k)}{\sin(\pi k)} \lim_{z \to k} f(z)\cos(\pi z) = f(k)
\end{aligned}
$$

Using the Residue Theorem: Part 2

Define the Contour S_n to be the following square.

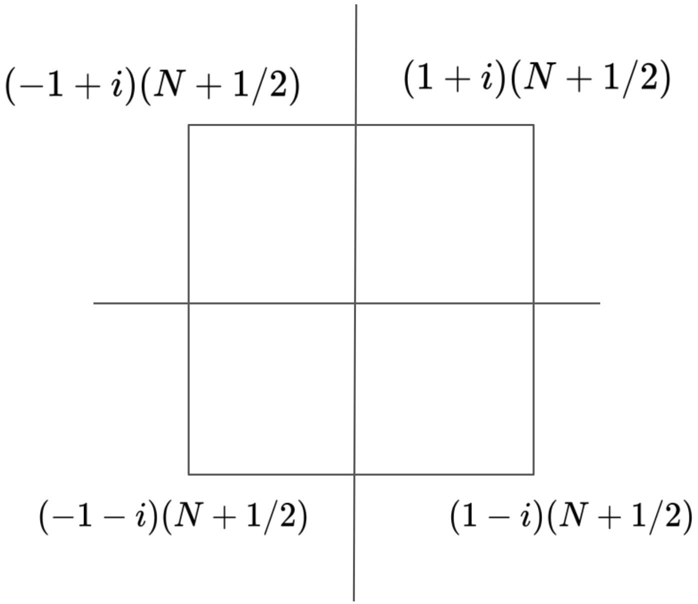
We wish to show that $\lim_{n\to\infty} \int_{S_n} g(z) = 0$

First, we have that for sufficiently large **N,** $\alpha \geq 2$

$$|f(z)| \leq |K/N^{\alpha}|$$

We also have:

$$\begin{aligned}|\cot(\pi z)| &= |1 + \tfrac{2}{e^{2\pi i z}-1}| \leq 1 + \tfrac{2}{|e^{2\pi i z}-1|} \\ &= 1 + \frac{2}{|e^{2\pi i(1+it)(N+1/2)}|} \leq 1 + 2/2 = 2\end{aligned}$$



Using the Residue Theorem: Part 4

Using our bounds, we have that

$$\left|\int_{S_n} g(z)dz\right| \leq \int_{S_n} \left|2\pi\frac{K}{N^\alpha}\right|dz = 2\pi\frac{K}{N^\alpha}\cdot 4(2n+1)$$

Which approaches 0 as N gets arbitrarily large.

However,

$$0 = \lim_{n\to\infty}\int_{S_n} g(z)dz = \lim_{n\to\infty} 2\pi i(\sum_{k=-N}^{N} f(k) + \sum \text{residues of g at singularities of f}$$

Therefore,

$$\sum_{n=-\infty}^{\infty} f(n) = -\sum \text{residues of g at singularities of f}$$

Using the Residue Theorem: Part 5

Finally, let $f(z) = 1/z^2$.

From the previous result, we have that

$$2\sum_{n=1}^{\infty} 1/n^2 = \sum_{n=-\infty}^{\infty} 1/n^2 = -Res(\frac{\pi\cot(\pi z)}{z^2}, 0)$$

However, $\cot(z) = \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} - \frac{2z^5}{945} - \frac{z^7}{4725} + \dots$

$$\frac{\pi\cot(\pi z)}{z^2} = \frac{1}{z^3} - \frac{\pi^2}{3z} - \frac{\pi^4 z}{45} + \dots$$

Therefore, $Res(\frac{\pi\cot(\pi z)}{z^2}, 0) = -\frac{\pi^2}{3}$

Finally, $\sum_{n=1}^{\infty} 1/n^2 = -\frac{1}{2}Res(\frac{\pi\cot(\pi z)}{z^2}, 0) = \frac{-1}{2}\frac{-\pi^2}{3} = \frac{\pi^2}{6}$ ∎

Summary

- Defined: **holomorphic, analytic, singularity, Laurent Series**, and **Residue.**
- Proof sketch of **Cauchy's Residue Theorem**
- Analyzed the function: $g(z) = \pi f(z) \cot(\pi z)$

$$Res(g(z), k) = f(k)$$

$$\lim_{n\to\infty} \int_{S_n} g(z) = 0$$

$$\sum_{n=-\infty}^{\infty} f(n) = -\sum \text{residues of g at singularities of f}$$

- By setting $f(z) = 1/z^2$ we showed $\sum_{n=1}^{\infty} 1/n^2 = \frac{\pi^2}{6}$

Source Reading

- Fundamentals of Complex Analysis for Mathematics, Science, and Engineering by Edward B. Saff, Arthur David Snider

Other works consulted

- Complex Analysis by Serge Lang