Using Residue Theory to Evaluate Infinite Sums

Zachary Star
Mentor: Keagan Callis
Directed Reading Program Summer 2018
Definitions

- A function $f : D \rightarrow \mathbb{C}$ is called **Holomorphic** if it is complex-differentiable in the domain $D$.

- A function $f : D \rightarrow \mathbb{C}$ is called **Analytic** if it has a convergent power series about some point $z_0 \in D$.

- A function has a **Singularity** at a point if it fails to be well defined at that point, or appears to “blow up” at that point.

We also use some well-known results from complex analysis.

$$e^{i\pi} + 1 = 0$$
Laurent Series: Generalizing Power Series in $\mathbb{C}$

- We are used to using Taylor series to describe a differentiable function as a convergent series.
- Suppose that we have a function analytic in some neighborhood around a point but not necessarily at that point.
- Then we write

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$
Complex Integration

Consider the problem of finding $\int_{C_r} (z - z_0)^n \, dz$ where $C$ is the closed contour around a circle given by $C_r : z(t) = z_0 + re^{it}$ for $t \in [0, 2\pi]$.

$$\int_{C_r} (z - z_0)^n \, dz = \int_0^{2\pi} r^n e^{int} ire^{it} \, dt = ir^{n+1} \int_0^{2\pi} e^{it(n+1)} \, dt$$

- For $n \neq -1$:
  $$ir^{n+1} \int_0^{2\pi} e^{it(n+1)} \, dt = \left. \frac{r^{n+1} e^{it(n+1)}}{(n+1)} \right|_0^{2\pi} = r^{n+1} \left( \frac{1}{n+1} - \frac{1}{n+1} \right) = 0$$

- For $n = -1$:
  $$ir^{n+1} \int_0^{2\pi} e^{it(n+1)} \, dt = it \bigg|_0^{2\pi} = 2\pi i$$
Residue Theory

We want an easier way to perform complex integration. Recall

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

Integrating both sides, we find that almost every term vanishes.

$$\int_C f(z) dz = \int_C a_0 + a_1 (z - z_0)^1 + a_{-1} (z - z_0)^{-1} + a_2 (z - z_0)^2 + a_{-2} (z - z_0)^{-2} + \ldots \, dz$$

$$= 0 + 0 + a_{-1} (2\pi i) + 0 + 0 + \ldots = 2\pi i a_{-1}$$

We therefore call the negative first coefficient in a function’s Laurent Series its Residue at $z_0$. 
Cauchy’s Residue Theorem

For a function having residues $z_0, z_1, \ldots, z_k$ inside a contour, we have the following formula.

$$\int_C f(z) \, dz = 2\pi i \sum_{j=0}^{k} \text{Res}(f, z_k)$$
Calculating Residues

We want to find a formula to calculate the residues of a function with the form:

\[ f(z) = a_{-m}(z - z_0)^{-m} + a_{-m+1}(z - z_0)^{-m+1} + \ldots + a_{-1}(z - z_0)^{-1} + a_0 + a_1(z - z_0) + \ldots \]

Multiplying by \((z - z_0)^m\), taking \(m - 1\) derivatives, and the limit approaching \(z_0\):

\[ ((z - z_0)^m f(z))^{(m-1)} = a_{-1}(m - 1)! + (m - 1)!a_0(z - z_0) + a_1(m - 1)!(z - z_0)^2 + \ldots \]

This gives us a formula in terms of \(m, z_0\), and derivatives of \(f(z)\)

\[ \frac{1}{(m-1)!} \lim_{z \to z_0} ((z - z_0)^m f(z))^{(m-1)} = \text{Res}(f(z), z_0) \]
Let \( g(z) = \pi f(z) \cot(\pi z) \), where \( f \) is a function decaying like \( 1/z^2 \).

We note that this function has singularities at all the integers, so we use our formula to calculate the residues at these points.

\[
\text{Res}(g, k) = \lim_{z \to k} (z - k) \pi f(z) \frac{\cos(\pi z)}{\sin(\pi z)}
\]

\[
= \lim_{z \to k} \frac{\pi (z - k)}{\sin(\pi k)} \lim_{z \to k} f(z) \cos(\pi z) = f(k)
\]
Using the Residue Theorem: Part 2

Define the Contour $S_n$ to be the following square.

We wish to show that $\lim_{n \to \infty} \int_{S_n} g(z) = 0$

First, we have that for sufficiently large $N$, $\alpha \geq 2$

$|f(z)| \leq |K/N^\alpha|$  

We also have:

$|\cot(\pi z)| = |1 + \frac{2}{e^{2\pi i z} - 1}| \leq 1 + \frac{2}{|e^{2\pi i z} - 1|}$

$= 1 + \frac{2}{|e^{2\pi i (1+it)(N+1/2)}|} \leq 1 + 2/2 = 2$
Using the Residue Theorem: Part 4

Using our bounds, we have that

$$\left| \int_{S_n} g(z) \, dz \right| \leq \int_{S_n} |2\pi \frac{K}{N^\alpha}| \, dz = 2\pi \frac{K}{N^\alpha} \cdot 4(2n + 1)$$

Which approaches 0 as N gets arbitrarily large.

However,

$$0 = \lim_{n \to \infty} \int_{S_n} g(z) \, dz = \lim_{n \to \infty} 2\pi i(\sum_{k=-N}^{N} f(k) + \sum \text{residues of } g \text{ at singularities of } f)$$

Therefore,

$$\sum_{n=-\infty}^{\infty} f(n) = -\sum \text{residues of } g \text{ at singularities of } f$$
Using the Residue Theorem: Part 5

Finally, let \( f(z) = 1/z^2 \).

From the previous result, we have that

\[
2 \sum_{n=1}^\infty \frac{1}{n^2} = \sum_{n=-\infty}^\infty \frac{1}{n^2} = -\text{Res}\left(\frac{\pi \cot(\pi z)}{z^2}, 0\right)
\]

However,

\[
\cot(z) = \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} - \frac{2z^5}{945} - \frac{z^7}{4725} + \ldots
\]

\[
\frac{\pi \cot(\pi z)}{z^2} = \frac{1}{z^3} - \frac{\pi^2}{3z} - \frac{\pi^4 z}{45} + \ldots
\]

Therefore,

\[
\text{Res}\left(\frac{\pi \cot(\pi z)}{z^2}, 0\right) = -\frac{\pi^2}{3}
\]

Finally,

\[
\sum_{n=1}^\infty \frac{1}{n^2} = -\frac{1}{2} \text{Res}\left(\frac{\pi \cot(\pi z)}{z^2}, 0\right) = -\frac{1}{2} \left(-\frac{\pi^2}{3}\right) = \frac{\pi^2}{6}
\]
Summary

- Defined: holomorphic, analytic, singularity, Laurent Series, and Residue.
- Proof sketch of Cauchy’s Residue Theorem
- Analyzed the function: \( g(z) = \pi f(z) \cot(\pi z) \)
  
  \[
  \text{Res}(g(z), k) = f(k) \\
  \lim_{n \to \infty} \int_{S_n} g(z) = 0 \\
  \sum_{n=-\infty}^{\infty} f(n) = -\sum \text{residues of } g \text{ at singularities of } f
  \]
- By setting \( f(z) = 1/z^2 \) we showed \( \sum_{n=1}^{\infty} 1/n^2 = \frac{\pi^2}{6} \)
Source Reading


Other works consulted

- Complex Analysis by Serge Lang