Thinking Mathematically

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“What is number theory about?”

- “Number theory is an excellent subject for learning the ways of mathematical thought. Every college student is familiar with basic properties of numbers, and yet the study of those familiar numbers leads us into waters of extreme depth…..”
- “Never should mathematics seem to be a mysterious collection of definitions, theorems, and proofs that arise from the void and must be memorized for a test.”

—Marshall, Odell, Starbird 2007

“So *this* is what division is?”

- “If we try to divide 1 by 2, then we seek an integer x such that $2x = 1$.” That’s what we actually mean by “division.”
- If a and m are both integers, then we could divide a by m and end up with another integer plus some remainder r that could be any number 0, 1, 2… all the way down to m - 1.
- **a = mq + r.**

“Are 1068 and 3054_7 the same number?”

$$\underline{3} \quad \underline{0} \quad \underline{5} \quad \underline{4}$$
$$7^3 \quad 7^2 \quad 7^1 \quad 7^0$$

$1068 = 7 * 152 + \mathbf{4}$

$152 = 7 * 21 + \mathbf{5}$

$21 = 7 * 3 + \mathbf{0}$

$3 = 7 * 0 + \mathbf{3}$

$1068 = 7(152) + 4$

$1068 = 7(7(21) + 5) + 4$

$1068 = 7(7(7(3) + 0\ ) + 5) + 4$

$1068 = 7(7(7(7(0) + 3) + 0) + 5) + 4)$

“How can these numbers even exist?!”

- “How many of the following can you find in $\mathbf{Z}_7$?”
- $4(5)$, $2-6$, $\frac{1}{2}$, $\frac{2}{5}$, $\sqrt{2}$, $\sqrt{-3}$, $\sqrt[3]{6}$
- $\sqrt{-3} \equiv \sqrt{4} \bmod 7$. What number times itself gives us 4?
- $\sqrt[3]{6}$: $1^3 = 1$, $2^3 = 8 \equiv 1 \bmod 7$, $3^3 = 27 \equiv 6 \bmod 7$
- I needed to think more deeply about the things I was asking the numbers to do.

Can you find an integer n > 1 *such that the sum 1 + ½ + … + 1/*n *is an integer?*

1	1/2	1/3	1/4	1/5	1/6	1/7	1/8	1/9	1/10
1	3/2	11/6	25/12	137/60	147/60	1089/420	2,283/ 840	1427/504	7387/ 2520

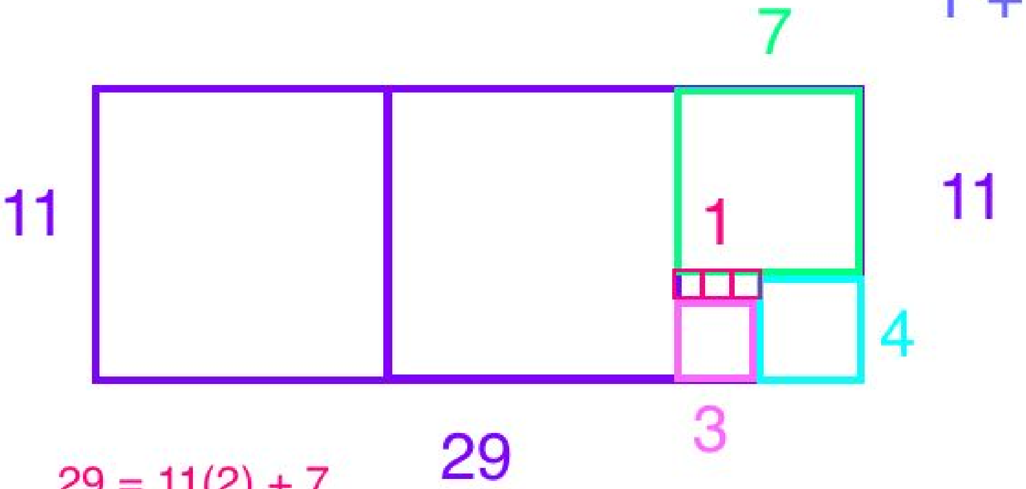
3/2	2/2	1/2						
11/6	6/6	3/6	2/6					
25/12	12/12	6/12	4/12	3/12				
137/60	60/60	30/60	20/60	15/60	12/60			
1,089/ 420	420/420	210/420	140/420	105/420	84/420	70/420	60/420	
2,283/ 840	840/840	420/840	280/840	210/840	168/840	140/840	120/840	105/840

2 1 1 1 3

0 1 2 3 5 8 29

1 0 1 1 2 3 11

$$2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{3}}}}$$



29 = 11(2) + 7
11 = 7(1) + 4
7 = 4(1) +3
4 = 3(1) + 1
3 = 1(3) + 0

"What are continued fractions trying to tell us about a number?"