A Little Lie (Theory) Never Hurt Anyone

Siddharth Taneja

in collaboration with

Arghya Sadhukhan

University of Maryland

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Introduction

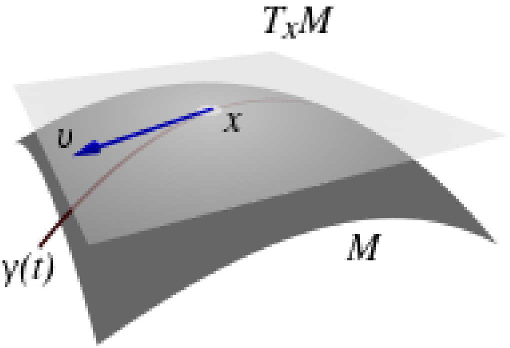
- Lie Algebra - A vector space $\mathfrak{g}$ with an **anti-symmetric**, **bilinear** product $(x, y) \mapsto [x, y]$ that satisfies the **Jacobi Identity**
  - $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$
- Examples
  - Any associative algebra (e.g. the set of all matrices) can be turned into a Lie algebra by defining $[x, y] := xy - yx$
  - $\mathbb{R}^3$ with the cross product, $[\vec{x}, \vec{y}] := \vec{x} \times \vec{y}$
- We will seek **representation** of these Lie Algebras: homomorphisms $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ for a chosen vector space V.
  - ρ is linear
  - $\rho([X, Y]) = [\rho(X), \rho(Y)] = \rho(X)\rho(Y) - \rho(Y)\rho(X)$
  - With any Lie algebra, along comes a 'free' representation, called the **adjoint representation** $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ given by $x \mapsto [x, \cdot]$; the fact that it's a representation follows from the Jacobi identity above.

Why Bother?

- They help us to understand a more "natural" object, the Lie Groups
- Lie Group - A group G that is also a differentiable manifold such that the group operation $(g, h) \mapsto g^{-1}h$ is smooth.
- Examples
  - $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{SL}_n(\mathbb{C})$, $\mathrm{SU}_n(\mathbb{C})$
  - From Physics we get the Lorentz, Poincaré, Symplectic $\mathrm{Sp}_{2n}(\mathbb{C})$ Groups, and E_8
- Lie Groups commonly encode symmetries
- Lie Algebras are **linearization** of Lie Groups, and they capture "local information" around the identity of the Lie Group. Since Lie Groups "look" the same near every other point (by translation), we get an idea of the group by studying the algebra.
  - Differential Geometry + Topology $\Rightarrow$ Linear Algebra + Abstract Algebra

Linearization?

- Let $\Psi_g(h) = g \cdot h \cdot g^{-1}$. Then define $\mathrm{Ad}(g) = (d\Psi_g)_e : T_eG \to T_eG$
- Then for the elements of T_eG, define $[x, y] := \mathrm{Ad}(x)(y)$
- $[\cdot, \cdot]$ defines a Lie Algebra structure on $\mathfrak{g} := T_eG$
- For example, this turns $\mathrm{SL}_n(\mathbb{C}) = \{M \in \mathrm{M}_n(\mathbb{C}) \mid \det(M) = 1\}$ into $\mathfrak{sl}_n(\mathbb{C}) = \{M \in \mathrm{M}_n(\mathbb{C}) \mid \mathrm{Tr}(M) = 1\}$



Some Caveats

- We have a few caveats before we go forward
  1. Multiple Lie Groups can correspond to the same Lie Algebra
     - Lie Groups which correspond to the same Lie Algebra are part of the same **isogeny** class

$$\begin{tikzcd} G_{\mathrm{SC}} \arrow[dr] & \\ \vdots \arrow[r] & \mathfrak{g} \\ G_{\mathrm{ad}} \arrow[ur] & \end{tikzcd}$$

     - $G_{\mathrm{ad}} = G_{\mathrm{SC}}/Z(G_{\mathrm{SC}})$
  2. We don't yet know how to go back from the Lie Algebra to the Lie Group for a given G in an isogeny class
     - i.e. $\mathfrak{g} \to \mathfrak{gl}_n \xrightarrow{?} G \to \mathrm{GL}_n$

Simplifying our study of Lie Algebras

- Abelian - $[x, y] = 0$ is boring, so we only focus on Lie algebras which have no **non-zero abelian ideals**
  - Things get ugly without this assumption
  - These are called the **semi-simple** Lie Algebras
  - One can classify these completely! (circa 1890)
- 2 nice features of these
  1. ad: $\mathfrak{g} \hookrightarrow \mathfrak{gl}(\mathfrak{g})$ since $\mathrm{Ker}(ad) = Z(\mathfrak{g}) = 0$ (the center of $\mathfrak{g}$).
     - So any semi-simple Lie Algebra is essentially a sub-algebra of matrices
  2. Every finite-dim representation of any such $\mathfrak{g}$ is **completely reducible**, so we need only focus on the "prime" representations
     - specifically those with no non-trivial $\mathfrak{g}$-invariant subspace

$\mathfrak{sl}_2(\mathbb{C})$

- We will study the simplest semi-simple Lie Algebra to understand how we will generalize
- $\mathfrak{sl}_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C}) | a + d = 0 \right\} = \mathrm{Tr}_2^{-1}(0)$
- $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
    - Check: $[h, x] = 2x$, $[h, y] = -2y$, and $[x, y] = h$
    - $\mathfrak{sl}_2(\mathbb{C}) = \mathbb{C}x \oplus \mathbb{C}y \oplus \mathbb{C}h$
    - h acts diagonally on any irreducible representation (V, ρ)
- So we write, $V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda$ where $V_\lambda = \{v \in V \mid \rho(h) \cdot v = \lambda v\}$
    - Call the weights $\mathcal{R} = \{\lambda \in \mathbb{C} \mid V_\lambda \neq 0\}$ (finite, since $\dim \mathfrak{sl}_2(\mathbb{C}) < \infty$)

$\mathfrak{sl}_2(\mathbb{C})$ continued

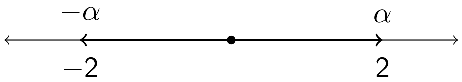
- Check: If $v \in V_\lambda$, then $x \cdot v \in V_{\lambda+2}$ and $y \cdot v \in V_{\lambda-2}$
- So $\mathcal{R}$ is an unbroken string of complex numbers separated by 2

$$\begin{tikzcd}
\lambda-4 \arrow[r, bend left, "x"] & \lambda-2 \arrow[l, bend left, "y"] \arrow[r, bend left, "x"] & \lambda \arrow[l, bend left, "y"] \arrow[r, bend left, "x"] & \lambda+2 \arrow[l, bend left, "y"] \arrow[r, bend left, "x"] & \lambda+4 \arrow[l, bend left, "y"]
\end{tikzcd}$$

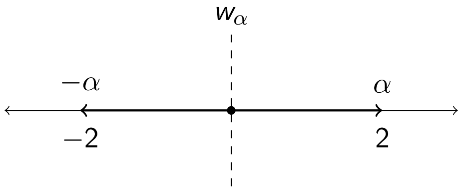
- Fact: $\mathcal{R} \subset \mathbb{Z}$, and $-\mathcal{R} = \mathcal{R}$ (symmetric about 0)
- So V is entirely determined by the **largest** (or smallest) element in $\mathcal{R}$
  - This is known as the **highest weight**: it is a **positive integer**
  - The corresponding eigenvector is the **highest weight vector**

Generalizations

- Fact: Every semi-simple $\mathfrak{g}$ has a maximal abelian subalgebra $\mathfrak{h}$ which acts diagonally on $\mathfrak{g}$
  - Known as the **Cartan subalgebra**
- Analogously, we get $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \mathcal{R} - \{0\}} \mathfrak{g}_\alpha = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathcal{R} - \{0\}} \mathfrak{g}_\alpha$
- for $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \forall H \in \mathfrak{h}, [H, X] = \alpha(H) \cdot X\}$, $\mathcal{R} = \{\alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \neq 0\}$
- These α's are called **roots**, with $\mathfrak{g}_\alpha$ being **root spaces**. These are one dimensional.
- One can show that, for any α, we have $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}$ and moreover $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \cong \mathfrak{sl}_2(\mathbb{C})$



Some properties



(i) $\mathcal{R}$ is finite, and spans $\mathfrak{h}^*$

(ii) $\forall \alpha \in \mathcal{R}, \exists$ a symmetry w_α that leaves $\mathcal{R}$ invariant, i.e. $w_\alpha(\beta) \in \mathcal{R}, \forall \beta \in \mathcal{R}$
- Reflection w.r.t the hyperplane perpendicular to α
- So in particular, if $\alpha \in \mathcal{R}$, then $w_\alpha(\alpha) = -\alpha \in \mathcal{R}$

(iii) $\forall \alpha \in \mathcal{R}$, $\mathcal{R} \cap \alpha\mathbb{C} = \{\pm\alpha\}$ (so the only multiples of a root which are also roots are the ones already predicted above)

(iv) $\forall \alpha, \beta \in \mathcal{R}$, $w_\alpha(\beta) - \beta \in \alpha\mathbb{Z}$

Possible Configurations

- These restrictions are pretty limiting, so we can classify them based on the dimension of $\mathfrak{h}^*$
- In 1D, we only get the above example
- in 2D, there are 4 possibilities

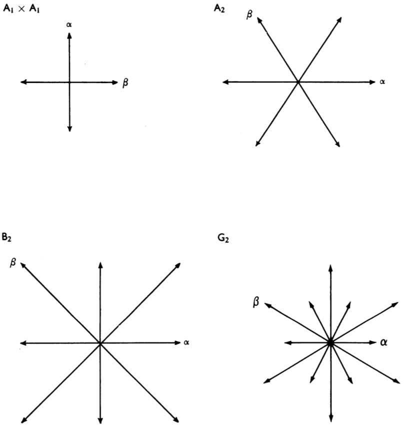
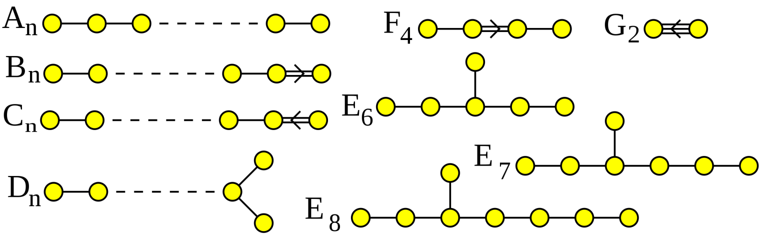


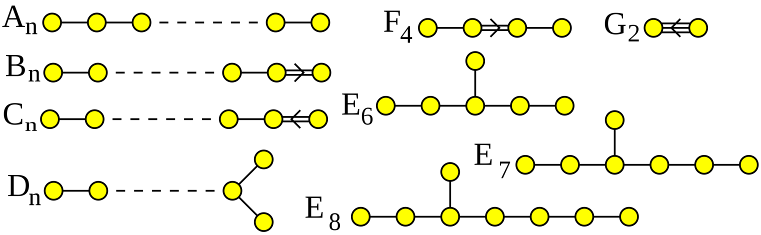
Figure 1

Dynkin Diagrams

- In general, given a set of root vectors, we can choose a basis $\{\alpha_1, \cdots, \alpha_\ell\}$ known as simple roots. We can show that $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle \in \{0, 1, 2, 3\}$ for $i \neq j$
- This motivates us to define the **Dynkin-Diagram** using the following rules
  1. Create ℓ nodes, one for each root
  2. Between each α_i and α_j, draw $k = \langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$ edges between them
  3. For each α_i and α_j, if $|\alpha_i| \neq |\alpha_j|$, add an arrow pointing to the shorter root



Dynkin Diagrams Cont.



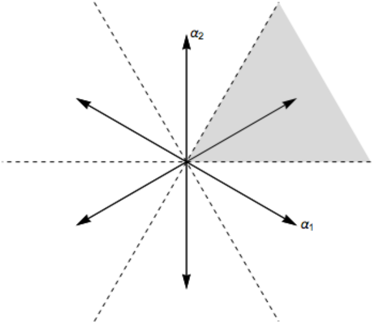
- The above diagram actually accounts for all possible root systems, in the following way
  - $A_n \longleftrightarrow \mathfrak{sl}_{n+1}$ for $n \geq 1$
  - $B_n \longleftrightarrow \mathfrak{so}_{2n+1}$ for $n \geq 2$
  - $C_n \longleftrightarrow \mathfrak{sp}_{2n}$ for $n \geq 3$
  - $D_n \longleftrightarrow \mathfrak{so}_{2n}$ for $n \geq 4$

What about Representations?

- In the $\mathfrak{sl}_2(\mathbb{C})$ picture, we found the possible set of weights was just $\mathbb{Z}$, with the highest weights being in $\mathbb{Z}^+$. The roots were $\{-2, 2\}$, so the group generated by the roots, $2\mathbb{Z}$, is a subset of the possible weights $\mathbb{Z}$.
- The same idea generalizes, as follows
- Given a $\mathfrak{g}$-rep V, write V in terms of $\mathfrak{h}$ actions:
    - $V = \bigoplus_{\lambda \in \pi(V)} V_\lambda, \pi(V) \subset \mathfrak{h}^*$ being the finite set of weights appearing in decomposition of V
- The set of all weights $\Lambda_W = \bigcup \pi(V)$ is a lattice in $\mathbb{R}^{\dim \mathfrak{h}}$, containing the lattice generated by the roots Λ_R
- It turns out that Λ_W / Λ_R is a finite group (For $\mathfrak{sl}_2(\mathbb{C})$, $\Lambda_W / \Lambda_R = \mathbb{Z}_2$)

Concluding

- It can be shown that for each finite-dimensional irreducible rep. of $\mathfrak{g}$ (up to iso), we can associate an element of Λ_W^+
  - $\Lambda_W^+ \subset \Lambda_W \subset \mathbb{R}^{\dim \mathfrak{h}}$
  - Λ_W^+ is the set of **dominant integral weights**
  - Λ_W^+ is a "cone" in the weight lattice



Summary

- Starting with Lie Groups, we can "linearize" to get Lie Algebras
- We care about representations, since they allow us to manipulate the group concretely
- We restrict to the "prime" (semi-simple) Lie Algebras, which have no abelian ideal
- By looking at the largest abelian subalgebra, we can decompose $\mathfrak{g}$ (or any rep V) into the simultaneous "eigenspaces"
- By looking at the "eigenvalues", we can solve the problem entirely geometrically, and therefore reduce to a full-classification of simple Lie Algebras, and codify these using the Dynkin Diagrams
- There is a $1-1$ correspondence between finite-dimensional irreducible representations of $\mathfrak{g}$ and the set Λ_W^+ of dominant integral weights