Weather model:

Questions to consider:

Given the probability distribution of the weather today is \([a, b, c]\)

- How do we predict the weather for tomorrow, if for each day, the probabilities of weather changes are all the same?
- Is it possible that after a thousand years, the chances of weather for each day remain unchanged?
Markov Chains - what is it?

- Formally, a Markov chain is defined to be a sequence of random variables \((X_n)_{n \geq 0}\), taking values in a set of states, which we denote by \(S\), with initial distribution \(\lambda\) and transition matrix \(P\), if
  - \(X_0\) has distribution \(\lambda = \{\lambda_i | i \in S\}\)
  - Transition matrix \(P = (p_{ij})_{i,j \in S}\), and the Markov property holds:
    \[ P(X_n = i_n | X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) = P(X_n = i_n | X_{n-1} = i_{n-1}) = p_{i_{n-1}i_n} \]
- Probability distributions

\[
P(X_n = j) = (\lambda P^n)_j
\]
\[
P_i(X_n = j) = P(X_{n+m} = j | X_m = j) = p_{ij}^{(n)}
\]
Markov Chains-communicating classes and irreducibility

We say that a state $i$ communicate with state $j$ if one can get to $i$ from $j$, as well as from $j$ to $i$ with only finite many evolution times. We denote this relation as $i \leftrightarrow j$.

Note: $i \rightarrow j$ if and only if $p_{i_1,j}, \ldots, p_{k_{n-1},j} > 0$. Also it requires the sequence $k_1, \ldots, k_{n-1}$ to be finite.

Also note that $i \leftrightarrow j$ means this relation is
(1) symmetric: if $i \rightarrow j$ then $j \rightarrow i$;
(2) reflective: $i \leftrightarrow i$;
(3) transitive: $i \leftrightarrow j$ and $j \leftrightarrow k$ imply $i \leftrightarrow k$. 
Markov Chains-communicating classes and irreducibility

The sets of states with states having such relation jointly are called communicating classes. Therefore we can partition the set $S$, into communicating classes with respect to this equivalence relation.

![Diagram](image)

Definition: A Markov chain is **irreducible** if its set of states $S$ is a single communicating class.
Markov Chains-communicating classes and irreducibility

Illustration of irreducible and reducible Markov chains:

Note: Irreducibility of a Markov chain prepares us to study the equilibrium state of this chain.
Markov Chains-aperiodicity of Markov chains

- Definition: A state i is called aperiodic, if there exists a positive integer N, such that \( p_{ii}^{(n)} > 0 \) for all \( n \geq N \).

- Theorem: If P is irreducible, and has an aperiodic state i, then for all states j and k, \( p_{jk}^{(n)} > 0 \) for all sufficiently large n. (therefore all states are aperiodic)

  Sketch of the proof:
  \[
p_{jk}^{(r+n+s)} = \sum_{i_1, \ldots, i_n} p_{ji_1}^{(r)} p_{i_1i_2} \cdots p_{i_{n-1}i_n} p_{i_nk}^{(s)} \geq p_{ji}^{(r)} p_{ii}^{(n)} p_{ik}^{(s)} > 0
  \]

- Definition: We call a Markov chain aperiodic if all its states are aperiodic.

Now, recall the question: after sufficiently large evolution times, will the distribution of states reach an equilibrium?
Markov Chains-Invariant distributions

- A measure on a Markov chain is any vector \( \lambda = \{ \lambda_i \geq 0 \mid i \in S \} \)
- In addition, \( \lambda \) is a distribution if \( \sum_{i \in S} \lambda_i = 1 \)
- We say a measure \( \lambda \) is invariant if \( \lambda = \lambda P \).

Theorem: Suppose that \((X_n)_{n \geq 0}\) is a Markov chain with transition matrix \(P\) and initial distribution \(\lambda\). If \(P\) is both irreducible and aperiodic, and has an invariant distribution \(\pi\), then

\[
P(X_n = j) = (\lambda P^n)_j \to \pi_j \text{ as } n \to \infty \text{ for all } j.
\]

In particular,

\[
p_{ij}^{(n)} \to \pi_j \text{ for all } i, j.
\]
Markov Chains-Invariant distributions

(picture credit to Seattle Refined)

(picture credit to smithsonian.com)

(picture credit to BBC NEWS)
Markov Chains-Invariant distributions

By assuming that the finite-state Markov chain is irreducible and aperiodic, we can apply the **Perron-Frobenius Theorem**.

- **The Perron-Frobenius Theorem:**
  - Let $A$ be a positive square matrix. Then
    - $A$ has one largest eigenvalue $\rho(A)$ in absolute value and it has an positive eigenvector.
    - $\rho(A)$ has geometric multiplicity 1.
    - $\rho(A)$ has algebraic multiplicity 1.

Note: Also hold for nonnegative $A$ s.t $A^m$ is positive after some power $m$.

By applying the Perron-Frobenius Theorem to $P$,

- $\pi P = \pi \iff \rho(P) = 1$ with unique positive left eigenvector $\pi$.
- All other eigenvalues are of absolute values $< 1$.

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Markov Chains - Invariant distributions

\[
\lim_{n \to \infty} q^n = \lim_{n \to \infty} \begin{bmatrix}
1 & * & \cdots & * \\
* & 1 & \cdots & * \\
\vdots & \ddots & \ddots & \ddots \\
* & \cdots & 1 & *
\end{bmatrix}^n
\begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\vdots \\
\lambda_n
\end{bmatrix}
\begin{bmatrix}
\pi_1 & \pi_2 & \cdots & \pi_n
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\pi_1 & \pi_2 & \cdots & \pi_n
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\pi_1 & \pi_2 & \cdots & \pi_n
\end{bmatrix}
\]
Markov Chains - Recurrence and transience.

- Let $(X_n)_{n \geq 0}$ be a Markov chain with transition matrix $P$. Then a state $i \in S$ is recurrent if

$$P_i(X_n = i \text{ for infinitely many } n) = 1$$

- We say that $i$ is transient if

$$P_i(X_n = i \text{ for infinitely many } n) = 0$$

Now we are ready to see one implementation of the abstract Markov chains - the random walks.
Simple random walks-one dimension
We start by studying simple random walk on the integer lattices. At each time step, the random walker flips a fair coin to decide its next move.

Let \( S_n \) denote the position at time \( n \), \( x \) be the position it starts at. At each time step \( j \),
\[
X_j = \begin{cases} 
1, & \text{if Head appears on the } j\text{-th throw;} \\
-1, & \text{otherwise.}
\end{cases}
\]

we have
\[
S_n = x + X_1 + \ldots + X_n
\]
\[
P(X_j = 1) = P(X_j = -1) = 1/2
\]

Questions:
• On average, how far is the walker from the starting point?
• Does the walker keeps returning to the origin or does it eventually leave forever?
Simple random walks-one dimension

It’s easy to check that
\[ E(S_n) = x + E(X_1) + \ldots + E(X_n) = x + 0 + \ldots + 0 = x; \]
and since (assume the walker starts from 0)
\[ \text{Var}(X) = E(X^2) - E(X)^2 = E(X^2) = 1 \]
we have
\[ \text{Var}(S_n) = 0 + \text{Var}(X_1) + \ldots + \text{Var}(X_n) = n \]
\[ \sigma_{S_n} = \sqrt{n} \] (typical distance from the origin)

✦ What does this inform to us?
In one dimension, there are at most \( \sqrt{n} \) integers that are within typical distance with the mean distance.

So the chance of lying on a particular integer should shrink as a constant times \( n^{-\frac{1}{2}} \).
\[ P(S_n = j) \sim \frac{C}{\sqrt{n}} \]

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Simple random walks-one dimension

We may notice that after an odd number of steps, the walker must end at an odd integer; similarly in order to get to an even integer, we need even steps.

So we claim that the return probability

\[ P(S_{2n} = 0) = \binom{2n}{n} \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^n = \frac{(2n)!}{n!n!} \left(\frac{1}{2}\right)^{2n} \]

Stirling’s formula states that as \( n \to \infty \),

\[ n! \sim \sqrt{2\pi n}^{n+\frac{1}{2}} e^{-n}. \]

Then

\[ P(S_{2n} = 0) = \frac{(2n)!}{n!n!} \left(\frac{1}{2}\right)^{2n} \sim \frac{\sqrt{2}}{\sqrt{2\pi n}^{1/2}} = \frac{C_0}{n^{1/2}}. \]
Simple random walks-one dimension

Define \( V \) to be a random variable that denotes the number of time the walker returns to 0, then

\[
V = \sum_{n=0}^{\infty} I\{S_{2n} = 0\}
\]

(\text{where } I\{A\} \text{ is an indicator function})

Consider the mean of the number of visits

\[
E(V) = \sum_{n=0}^{\infty} E(I\{S_{2n} = 0\}) = 1 + \sum_{n=1}^{\infty} P(S_{2n} = 0) = 1 + \sum_{n=1}^{\infty} \frac{\sqrt{2}}{\sqrt{2\pi}} n^{-\frac{1}{2}}
\]

\[
= 1 + \frac{\sqrt{2}}{\sqrt{2\pi}} \sum_{n=1}^{\infty} n^{-\frac{1}{2}} = \infty
\]

(Recall that the sum \( \sum_{n=1}^{\infty} n^{-\frac{1}{2}} \) diverges since \( \frac{1}{2} < 1 \).)

If we let \( q = P(\text{the walker ever return to 0}) \), then we can show that \( q = 1 \) by supposing \( q < 1 \), and draw contradiction that \( E(V) \) will actually be finite.
Simple random walks-higher dimensions

• What will happen if the random walker takes action in higher dimensions, say $Z^d$?
  • In each direction, the random walks will be performed as in one dimension.
  • In $2n$ steps, we expect $(2n/d)$ steps to be taken in each of the $d$-directions.

$$P(\text{any particular integer}) \sim \frac{c_d}{n^{d/2}}$$

• Return to origin:

Since $P(S_n = 0) \sim \frac{c_d}{n^{d/2}}$

$$E(V) = \sum_{2n=0}^{\infty} P(S_n = 0) \sim \sum_{n=0}^{\infty} \frac{c_d}{n^{d/2}} = \begin{cases} < \infty, d \geq 3 \\ = \infty, d = 1,2 \end{cases}$$

• The results correspond to the facts that if the Markov chain is a simple symmetric on $Z^2$, all states are recurrent; if it’s on $Z^d$, $d \geq 3$, all states are transient.
References:

❖ Cameron, M. (n.d.). *Discrete time Markov chains.*
