The power of Nets!

When sequences are not enough

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Outline

Nets

Definition

A net is a function $P : \Lambda \to X$, where Λ is a directed set and X is an arbitrary set. We denote $P(\lambda)$ by x_λ, and the net by $(x_\lambda)_{\lambda \in \Lambda}$.

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Example

The natural numbers with their usual order operation.

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Definition

A net (x_λ) in X converges to $x \in X$ if for each neighborhood U of x, there is a $\lambda_0 \in \Lambda$ so that $\lambda \geq \lambda_0$ implies x_λ is in U. Equivalently, the net is enventually in every neighborhood of x.

Product spaces

Theorem

A net (x_λ) in a product $\prod_{\alpha \in \Gamma} X_\alpha$ converges to $x \in X$ if and only if for each $\alpha \in \Gamma$, $J_\alpha(x_\lambda) \to J_\alpha(x)$ in X_α.

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Compact spaces

For many spaces sequences are enough to characterize compactness; this is usually presented as sequential compactness is equivalent to compactness in most "nice" spaces. There are counterexamples to that statement, however,in more general settings. All is not lost, as can be seen from the next theorem.

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Theorem

A space X is compact if and only if every ultranet converges in X.

Continuous maps

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Lemma

The projection maps $J_\alpha : \prod_{\alpha \in \Gamma} X_\alpha \to X_\alpha$*, are continuous for each* α*.*

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Each of the above hold in general, and do not require the use of nets.

Tychonoff

Theorem (Tychonoff)

The non-empty product $X = \prod_{\alpha \in \Gamma} X_\alpha$ is compact if and only if each factor is compact.

The original proof of this theorem did not use nets and is much harder to prove. The proof we give below hides this difficulty in the notion of ultranets, and is only four lines.

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Now assume that each factor, X_α, is compact and let $(x_\lambda)_{\lambda \in \Lambda}$ be an ultranet in $X = \prod_{\alpha \in \Gamma} X_\alpha$. Then for each α, $(J_\alpha(x_\lambda))$ is an ultranet in X_α and hence converges, as each factor is compact. This says that (x_λ) converges in X by the previous theorem. Finally, X is compact since every ultranet in X converges. $\square$

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- Nets take on the role of sequences when the spaces become more complicated, e.q. spaces without a metric.
- The notion of an ultra-net characterizes compactness in a more general setting that sequential compactness can account for.
- The Tychonoff Theorem states that an arbitrary product of compact spaces is again compact.

For Further Reading I

S. Willard.
General Topology.
Dover Books, 1970.